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ON MELTING AND FREEZING FOR THE 2D RADIAL STEFAN PROBLEM

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Abstract. We consider the two dimensional free boundary Stefan problem describing the evolution of a spherically symmetric ice ball $\{r \le \lambda(t)\}$. We revisit the pioneering analysis of [20] and prove the existence in the radial class of finite time *melting* regimes

$$\lambda(t) = \left\{ \begin{array}{ll} (T-t)^{1/2} e^{-\frac{\sqrt{2}}{2}\sqrt{|\log(T-t)|}+O(1)} & \\ (c+o(1))\frac{(T-t)^{\frac{k+1}{2}}}{|\log(T-t)|^{\frac{k+1}{2k}}}, \quad k \in \mathbb{N}^* & \end{array} \right. \quad \text{as } t \to T$$

which respectively correspond to the fundamental *stable* melting rate, and a sequence of codimension $k \in \mathbb{N}^*$ excited regimes. Our analysis fully revisits a related construction for the harmonic heat flow in [42] by introducing a new and canonical functional framework for the study of type II (i.e. non self similar) blow up. We also show a deep duality between the construction of the melting regimes and the derivation of a discrete sequence of global-in-time *freezing* regimes

$$\lambda_\infty - \lambda(t) \sim \left\{ \begin{array}{l} \frac{1}{\log t} \\ \frac{1}{t^k(\log t)^2}, \quad k \in \mathbb{N}^* \end{array} \right. \quad \text{as } t \to +\infty$$

which correspond respectively to the fundamental *stable* freezing rate, and excited regimes which are codimension k stable.

1. Introduction

1.1. **Setting of the problem.** We consider the classical two dimensional one-phase Stefan problem on an *external* domain. The unknowns are the moving domain $\Omega(t) \subset \mathbb{R}^2$ and the temperature function $u : \Omega(t) \to \mathbb{R}$ which evolve according to:

$$\left\{ \begin{array}{l} \partial_t u - \Delta u = 0 \ \text{ in } \ \Omega(t) \\ \partial_n u = V_{\partial\Omega(t)} \ \text{ on } \ \partial\Omega(t) \\ u = 0 \ \text{ on } \ \partial\Omega(t) \end{array} \right. \tag{1.1}$$

where $V_{\partial\Omega(t)}$ stands for the normal velocity of the moving boundary $\partial\Omega(t)$[1]. The temperature u may either be assumed to be positive initially in $\Omega(0)$, in which case the maximum principle and the Dirichlet boundary condition ensure that it will remain positive in $\Omega(t)$, or on the contrary the data may be undercooled with initially non positive temperature in some regions in space. The cavity represents a circular block of ice kept at constant temperature $u = 0$. If the cavity vanishes at a later time we refer to this process as *melting* and if it expands, we refer to it as *freezing*.

[1]For any given parametrization $\gamma(t,\cdot) : \mathbb{R} \to \mathbb{R}^2$ of $\partial\Omega(t)$, the normal velocity is given by the formula $V_{\partial\Omega(t)} = \partial_t\gamma \cdot n$, where n stands for the outward pointing unit normal with respect to $\partial\Omega(t)$.

1.2. **Cauchy theory.** There exists a large number of works pertaining to the questions of existence, uniqueness, regularity, and well-posedness for the classical Stefan problem. *Weak* solutions were first defined and shown to exist in [22] and their properties were further studied in many works, see [9, 1, 2, 10, 11, 46] and references therein. The classical Stefan problem lends itself to a different notion of a weak solution, the so-called *viscosity* solutions. For an overview of the seminal works on the regularity theory for such solutions, we point the reader to [3] and references therein, and for the basic existence and uniqueness theory to [23, 24]. In the class of *classical* solutions, first existence results are traced back to [18, 30]. For well-posedness results in energy-based Sobolev-type spaces see [16, 17] and in L^p-type spaces [12, 39].

In this article, we work in a radially symmetric situation and therefore the Cauchy theory is simpler. The domain $\Omega(t)$ is given by

$$\Omega(t) = \{x \in \mathbb{R}^2;\ |x| \geq \lambda(t)\},$$

and the Cauchy problem (1.1) becomes:

$$\begin{cases} u_t - u_{rr} - \frac{1}{r}u_r = 0 \ \text{ in } \ \Omega(t) \\ u_r(t, \lambda(t)) = -\dot{\lambda}(t) \\ u(t, \lambda(t)) = 0 \\ u(0, \cdot) = u_0, \ \lambda(0) = \lambda_0. \end{cases} \tag{1.2}$$

It is well posed in H^2: for all $(u_0, \lambda_0) \in H^2 \times \mathbb{R}_+^*$ with u_0 radially symmetric, there exists a unique solution $(u(t), \lambda(t)) \in \mathcal{C}([0,T), H^2(\Omega)) \times \mathcal{C}^1([0,T), \mathbb{R}_+^*)$ to (1.2), and

$$T < +\infty \ \text{ implies } \ \left(\lim_{t\to T} \|u(t)\|_{H^2(\Omega(t))} = +\infty \ \text{ or } \ \lim_{t \to T} \lambda(t) = 0 \right).$$

We recall the classical proof[2] in Appendix D. A simple integration by parts using the boundary conditions (1.2), see (D.4), ensures the uniform control of the Dirichlet energy:

$$\int_{|x|\geq \lambda(t)} |\nabla u(t,x)|^2 dx \leq \int_{|x| \geq \lambda(0)} |\nabla u_0(x)|^2 dx \tag{1.3}$$

in the melting regime $\dot{\lambda} < 0$.

1.3. **Main results.** The main result of this paper is the existence of a discrete sequence of melting rates with k nonlinear instability directions, $k \in \mathbb{N}$.

Theorem 1.1. *[Melting dynamics] There exists a set of data (u_0, λ_0) in $H^2 \times \mathbb{R}_+^*$, with u_0 arbitrarily small in $\dot{H}^1$, such that the corresponding solution $(u, \lambda) \in \mathcal{C}([0,T), H^2) \times \mathcal{C}^1([0,T), \mathbb{R}_+)$ to the exterior Stefan problem (1.2) melts in finite time $0 \leq T = T(u_0, l_0) < \infty$ with the following asymptotics:*
<u>1. Stable regime</u>*: the fundamental rate is given by*

$$\lambda(t) = (T-t)^{1/2} e^{-\frac{\sqrt{2}}{2}\sqrt{|\log(T-t)| + O_{t\to T}(1)}} \tag{1.4}$$

and is stable *by small radial perturbations in H^2; this regime corresponds to positive data $u_0 > 0$.*
<u>2. Excited regimes</u>*: the excited melting rates are given by*

$$\lambda(t) = (c^*(u_0, \lambda_0) + o_{t\to T}(1)) \frac{(T-t)^{\frac{k+1}{2}}}{|\log(T-t)|^{\frac{k+1}{2k}}}, \quad k \in \mathbb{N}^*, \tag{1.5}$$

[2] see for instance [19, 8] and references therein for a Cauchy theory in the class of Hölder spaces.

for some $c^*(u_0, \lambda_0) > 0$; it corresponds to undercooled initial data lying on a locally Lipchitz H^2 manifold of codimension k.
3. Non concentration of energy*: In all cases, there exists $u^* \in \dot{H}^1$ such that*

$$\lim_{t \to T} \|\nabla u \chi_{\{|x| \geq \lambda(t)\}} - \nabla u^*\|_{L^2(\mathbb{R}^2)} = 0. \tag{1.6}$$

Comments on the result.

1. Previous works. Let us recall that a melting scenario with asymptotics on the melting rate is first proposed in the pioneering work [20] [3]. The analysis involves first a change of variables similar to [21], the meaning of which for the Stefan problem is not obvious, and a complicated matching procedure. In a series of recent works on both energy critical and supercritical parabolic and dispersive problems [33, 41, 42, 43, 44, 4], a different angle of attack for the construction of type II blow up bubbles is proposed. The strategy consists of two steps: construction of a high order approximate solution based on the expansion of the blow up profile with respect to a well chosen small parameter, and high order Sobolev energy estimates which allow for sufficient decay to close the estimates. The strength of this approach lies in its robustness as it in particular applies to Schrödinger and wave problems which are more delicate to handle. Its weakness however is that it requires the control of a large number of derivatives to get sufficient decay which may not be sharp in the parabolic setting.

2. Role of the dimension. This paper deals with the case $d = 2$, which is the energy critical case, but the higher dimensions $d \geq 3$ could be treated by an entirely analogous approach. In fact, the case $d = 2$ is the most complicated case, displaying small logarithmic gains only and strong coupling between the various components of the solution[4].

The main novelty of the proof of Theorem 1.1 is the derivation of a sharp functional setting for the construction of type II, i.e. non self similar blow up bubbles, here applied to a melting problem. Our analysis relies on new weighted energy bounds with a degenerate Gaussian weight based on the spectral decomposition of the leading order linear operator after renormalization. This is conceptually a continuation of the Giga-Kohn approach [14] to the type I blow up in the energy subcritical range. Our new set of estimates simplifies both the derivation of the approximate solution and the closure of the nonlinear energy bounds by using in an optimal way the dissipative structure of the problem, see in particular (3.48). The existence of a degenerate resonance leading the blow up rate is reminiscent of the derivation of the celebrated "log-log law" for the mass critical nonlinear Schrödinger equation, see [27, 38, 32, 31]. A recent series of works by Merle and Zaag [35, 36, 37] suggest that this approach may be of great interest for dispersive wave like problems as well.

Moreover, our approach applies as well to the construction of solutions that asymptotically converge to the solitary wave ($u = 0, \lambda = \text{const} > 0$), here meaning freezing, and the proof exhibits a deep duality for the derivation of the melting/freezing rates.

[3] A different version of (1.4) is computed in [20], but a correction is suggested in [19]. A similar problem occurs in [21], see the correct law in [44].

[4] see for example (3.32).

Theorem 1.2. *[Freezing dynamics] There exists a set of data* (u_0, λ_0) *in* $H^2 \times \mathbb{R}_+^*$, *arbitrarily small in* $\dot{H}^1$, *such that the corresponding solution* $(u, \lambda) \in \mathcal{C}([0,T), H^2) \times \mathcal{C}^1([0,T), \mathbb{R}_+)$ *to the exterior Stefan problem (1.2) exists globally-in-time. Furthermore, let*

$$\lambda_\infty = \sqrt{\lambda_0^2 - \frac{1}{\pi}\int_{\Omega_0} u_0(x)\, dx}, \tag{1.7}$$

then the solution exhibits an asymptotic freezing

$$\lim_{t\to+\infty} \lambda(t) = \lambda_\infty > 0$$

with the following asymptotics:
1. Stable regime*: the fundamental freezing rate is given by*

$$\lambda_\infty - \lambda(t) = \frac{c(u_0,\lambda_0)(1 + o_{t\to+\infty}(1))}{\log t} \tag{1.8}$$

for some $c(u_0, \lambda_0) > 0$*; it is* stable *with respect to small well localized smooth radial perturbations; this regime contains negative data* $u_0 < 0$.
2. Excited regimes*: excited melting rates are given by*

$$\lambda_\infty - \lambda(t) = \frac{c_k(u_0,\lambda_0)(1 + o_{t\to+\infty}(1))}{t^k(\log t)^2}, \quad k \in \mathbb{N}^*, \tag{1.9}$$

for some $c_k(u_0, \lambda_0) > 0$ *and correspond to superheated well localized initial data lying on a locally Lipchitz manifold of codimension* k *in some well localized norm.*
3. Energy asymptotics*: in all cases, the Dirichlet energy dissipates at the rate*

$$\|\nabla u(t)\|_{L^2(\Omega(t))} = \frac{d_k(u_0,\lambda_0)(1 + o_{t\to+\infty}(1))}{t^{k+1}\log t}, \quad k \in \mathbb{N}, \tag{1.10}$$

for some $d_k(u_0, \lambda_0) > 0$.

Comments on the result.

1. More melting regimes. In the setting of the Stefan and Hele-Shaw problems, the authors consider in [40] a melting scenario for the one-phase Stefan problem outside a fixed domain containing the origin and kept at a pre-fixed non-negative and nontrivial temperature, acting as an effective heat source. The liquid thus expands to infinity for positive initial data and an asymptotic rate of expansion for the free-boundary radius is obtained. Note that this situation is quite different from our setting as there is no such heat source in our case, and the freezing/melting process is driven entirely by the choice of initial conditions.

2. Solitary wave regimes. A non trivial global-in-time dynamics with convergence to the solitary wave similar to that described by theorem 1.2 has been derived in other critical settings, see for example [15], [28], [29]. The quantized convergence rates with logarithmics corrections are reminiscent of some classical nonlinear dynamical systems scenarios, see for example [13].

The first main open problem following this work is the understanding of the full non radial stability of the free boundary problem in the stable melting regime $k = 0$ which should be amenable to our approach. Let us mention that a related problem in the context of evaporating drops was recently studied and solved in the setting of a *self-similar* collapse in the very nice work [7]. The second main open problem is to give a complete description of the flow for small initial data, and here the

constructions and underlying functional framework of Theorem 1.1 and Theorem 1.2 will be essential steps.

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Notations. We denote the ball of radius K by

$$B_K(\mathbb{R}^d) := \{x \in \mathbb{R}^d, |x| \leq K\}$$

and set

$$\Lambda := y\partial_y.$$

For any $\alpha \geq 0$ we denote the external domain:

$$\Omega_\alpha := \{x \in \mathbb{R}^2 \big| \ |x| \geq \alpha\}.$$

When $\alpha = 1$, we shall simply write $\Omega_1 = \Omega$. We define the weight

$$\rho_\pm(z) := e^{\pm\frac{|z|^2}{2}}$$

and the scalar product on $\Omega_{\sqrt{b}}$ by

$$\langle f, g\rangle_{b,\pm} = \int_{\sqrt{b}}^\infty fg\rho_\pm z dz$$

and the associated norms

$$\|u\|_{L^2_{\rho,b,\pm}} = \left(\int_{\sqrt{b}}^\infty u^2\rho_\pm z dz\right)^{\frac{1}{2}}, \quad \|u\|_{H^1_{\rho,b,\pm}} = \left(\|\partial_z u\|^2_{L^2_{\rho,b,\pm}} + \|u\|^2_{L^2_{\rho,b,\pm}}\right)^{\frac{1}{2}}$$

and

$$\|u\|_{H^2_{\rho,b,\pm}} = \left(\|\Delta u\|_{L^2_{\rho,b,\pm}} + \|\partial_z u\|^2_{L^2_{\rho,b,\pm}} + \|u\|^2_{L^2_{\rho,b,\pm}}\right)^{\frac{1}{2}}.$$

We define for $b > 0$ the Hilbert space

$$H^1_{\rho,b,\pm} = \{u : \Omega_{\sqrt{b}} \to \mathbb{R}, \ \ u \text{ radial with } \ \|u\|_{H^1_{\rho,b,\pm}} < +\infty \ \text{ and } \ u(\sqrt{b}) = 0\} \tag{1.11}$$

equipped with the scalar product $\langle\cdot,\cdot\rangle_{b,\pm}$, and for $b = 0$:

$$H^1_{\rho,0,\pm} = \{u : \mathbb{R}^2 \to \mathbb{R}, \ \ u \text{ radial with } \ \|u\|_{H^1_{\rho,0,\pm}} < +\infty\}$$

equipped[5] with the scalar product $\langle\cdot,\cdot\rangle_0$. Similarily, we define the renormalized quantities:

$$(f, g)_{b,\pm} = \int_1^\infty fg\rho_{b,\pm} y dy, \quad \rho_{b,\pm} = e^{\pm\frac{b|y|^2}{2}}$$

and the norms

$$\|v\|_{b,\pm} = \left(\int_1^\infty v^2\rho_{b,\pm} y dy\right)^{\frac{1}{2}}, \ \|v\|_{H^1_{b,\pm}} = \|v\|_{b,\pm} + \|\partial_y v\|_{b,\pm}, \ \|v\|_{H^2_{b,\pm}} = \|v\|_{H^1_{b,\pm}} + \|\Delta v\|_{b,\pm},$$

and the Hilbert space

$$H^1_{b,\pm} = \{v : \Omega \to \mathbb{R}, \ \ v \text{ radial with } \ \|v\|_{H^1_{b,\pm}} < +\infty \ \text{ and } \ v(1) = 0\}. \tag{1.12}$$

[5] Observe that $u(\sqrt{b}) = 0$ ensures that $H^1_{\rho,b}$ can be isometrically embedded into $H^1_{\rho,0}$ by considering the map $u \mapsto u\mathbf{1}_{z\geq\sqrt{b}}$.

We define the sequence of numbers:

$$\alpha_0 := 0, \quad \alpha_j := \sum_{i=0}^{j-1} \frac{1}{j-i} \quad \text{for} \quad j \geq 1. \tag{1.13}$$

Throughout the paper, summations over $0 \leq j \leq k-1$ are empty for $k = 0$.

1.4. **Strategy of the proof.** Problem (1.2) is invariant under an energy critical scaling: if (u, λ) solves (1.2), then so does

$$u_\mu(t,r) := u(\mu^2 t, \mu r), \quad \lambda_\mu(t) := \frac{\lambda(t)}{\mu}, \tag{1.14}$$

and the scaling leaves the Dirichlet energy[6] unchanged. We therefore renormalize the flow

$$u(t,r) = v(s,y), \quad \frac{ds}{dt} = \frac{1}{\lambda^2(t)}, \quad y = \frac{x}{\lambda(t)}, \tag{1.15}$$

and obtain the renormalized equation with a *fixed* boundary:

$$\begin{cases} \partial_s v - \frac{\lambda_s}{\lambda}\Lambda v - \Delta v = 0, \quad y > 1; \\ v(s,1) = 0 \\ v_y(s,1) = -\frac{\lambda_s}{\lambda} \end{cases} \tag{1.16}$$

step 1 Perturbative spectral analysis. We start with the description of melting regimes. Let in a first approximation

$$b = -\frac{\lambda_s}{\lambda}, \quad 0 < b \ll 1,$$

then the linear operator driving (1.16) is

$$\mathcal{H}_b = -\Delta + b\Lambda, \quad v(1) = 0.$$

Our first new input is to diagonalize this operator in a suitable Hilbert space. Indeed, $\mathcal{H}_b$ is self adjoint with respect to the measure $e^{-\frac{b|y|^2}{2}}dy$, and has up to a shift compact resolvent in the Hilbert space $H^1_{b,-}$, and hence discrete spectrum. However, the limit $b \to 0$ is singular in the sense that the limiting operator is the Laplacian with resonant eigenmodes and continuous spectrum. After renormalization, we may equivalently consider

$$H_b = -\Delta + \Lambda, \quad v(\sqrt{b}) = 0$$

which formally is a deformation of the standard harmonic oscillator, but with a singular boundary condition. We claim that in these renormalized variables, the operator H_b can be diagonalized using a perturbative Lyapounov-Schmidt type argument in the weighted Hilbert space $H^1_{\rho,b,-}$. The first k eigenvalues are given for $0 < b < b^*(k)$ by

$$\lambda_{b,k} = 2k + \frac{2}{|\log b|} + o\left(\frac{1}{|\log b|}\right), \quad k \in \mathbb{N}$$

with a corresponding asymptotic expansion of the eigenmodes, see Proposition 2.3. After renormalization, this produces a family of eigenvectors

$$\mathcal{H}_b \eta_{b,k} = b\lambda_{b,k}\eta_{b,k}, \quad \eta_{b,k}(1) = 0.$$

step 2 Approximate solution and modulation equations. We now freeze

$$-\frac{\lambda_s}{\lambda} = a, \quad 0 < a \ll 1 \tag{1.17}$$

[6] which is dissipated from (1.3) in the melting regime.

and rewrite the renormalized flow

$$\partial_s v + \mathcal{H}_b v + (a-b)\Lambda v = 0, \quad v(s,1) = 0, \quad \partial_y v(s,1) = a$$

for a parameter b which will be chosen later. We look for an approximate solution of the form

$$Q_\beta(s,y) = \sum_{j=0}^{k} b_j(s)\eta_{b(s),j}(y).$$

After projection onto each eigenmode, we obtain the leading order dynamical system:

$$\left\{ \begin{array}{l} (b_j)_s + bb_j\lambda_{b,k} + \frac{2(a-b)b_j}{|\log b|} + \frac{jb_j}{b}\Phi = 0, \quad 0 \le j \le k \\ \Phi = b_s + 2b(a-b). \end{array} \right. \tag{1.18}$$

This sytem is complemented by the renormalized nonlinear free boundary conditon $\partial_y v(1) = a$ which forces the leading order relationship

$$a = \sum_{j=0}^{k} b_j \left(1 + \frac{2\alpha_j}{|\log b|}\right) + \text{lot}$$

with $(\alpha_j)_{0\le j\le k}$ given by (1.13). It remains to chose $b(a)$ which is done through the choice

$$\Phi = 0 \tag{1.19}$$

which will be motivated below. If b_k dominates over the remaining b_j-s, $j = 0, \dots, k-1$, the integration of the dynamical system

$$\left\{ \begin{array}{l} (b_k)_s + \left(2k + \frac{2}{|\log b|}\right) bb_k + \frac{2(a-b)b_k}{|\log b|} = 0 \\ b_s + 2b(a-b) = 0 \\ a = b_k\left(1 + \frac{2\alpha_k}{|\log b|}\right) \\ \frac{ds}{dt} = \frac{1}{\lambda^2}, \quad -\frac{\lambda_s}{\lambda} = a \end{array} \right. \tag{1.20}$$

leads to finite time melting with the rate (1.4) for $k = 0$ and (1.5) for $k \ge 1$, which as solutions to (1.20) display k instability directions.

step 3 Energy estimate. We now construct a solution of the form

$$v(s,y) = \sum_{j=0}^{k} b_j(s)\eta_{b(s),j}(y) + \varepsilon(s,y)$$

with

$$(\varepsilon, \eta_{b,j}) = 0, \quad 1 \le j \le k \tag{1.21}$$

and close an energy estimate for the remainder ε. Here we note that for a solution to the linear problem

$$\partial_s\varepsilon + \mathcal{H}_b\varepsilon + (a-b)\Lambda\varepsilon = 0,$$

the time dependance of $b(s)$ yields a modified energy identity

$$\frac{1}{2}\frac{d}{ds}\int \varepsilon^2 e^{-\frac{b|y|^2}{2}}dy = -(\mathcal{H}_b\varepsilon,\varepsilon) + \underbrace{(b_s + 2b(a-b))}_{=\Phi}\int |y|^2\varepsilon^2 e^{-\frac{b|y|^2}{2}}dy$$

and hence the choice of b (1.19) to cancel the second term in the energy identity[7]. To the leading order, thanks to the orthogonality conditions (1.21) and the spectral

[7] which involves a different type of norm for which the spectral gap constant is not explicit and would thus lead to severe difficulties.

gap estimate in weighted spaces associated to the knowledge of the kernel of $\mathcal{H}_b$, we obtain the fundamental energy estimate:

$$\frac{1}{2}\frac{d}{ds}\int \varepsilon^2 e^{-\frac{b|y|^2}{2}}dy = -(\mathcal{H}_b\varepsilon,\varepsilon) \leq -(2k+2)b\int \varepsilon^2 e^{-\frac{b|y|^2}{2}}dy.$$

An integration-in-time will produce the necessary decay to close the bound on the radiative part of the solution, i.e. ε. The situation is however more complicated since the problem cannot close at the level of H^1 Sobolev regularity, and instead forces us to take one more derivative. However, at the H^2 level the corresponding energy identity produces dangerous boundary terms. These come with a particular structure and may beautifully enough be handled through time integration[8], see Proposition 3.10. This part of the analysis is a replacement for the polynomially weighted estimates in [42], and uses in an optimal way the dissipative structure of the equation[9] and the nonlinear algebra induced by the free boundary.
The construction of the manifold of initial data to avoid the codimension k instabilities of the system of ODE's (1.18) is finally performed using a now classical Brouwer type argument as in [5, 4, 44, 34].

step 4 Freezing. These regimes correspond to an expansion of the circular ice block, reflected in a change of sign in (1.16), (1.17):

$$\frac{\lambda_s}{\lambda} = A > 0.$$

This causes a modification in the spectrum of the operator

$$H_B = -\Delta - B\Lambda, \quad v(\sqrt{B}) = 0, \quad B > 0,$$

which admits shifted eigenvalues:

$$\lambda_{B,k} = 2k + 2 + \frac{2}{|\log B|} + o\left(\frac{1}{|\log B|}\right), \quad k \in \mathbb{N}.$$

Computations parallel to the melting case lead to the dynamical system

$$\begin{cases} (B_k)_s + \left(2k + \frac{2}{|\log b|}\right)BB_k + \frac{2(B-A)B_k}{|\log B|} = 0 \\ B_s + 2B(B-A) = 0 \\ A = B_k\left(1 + \frac{2\alpha_k}{|\log B|}\right) \\ \frac{ds}{dt} = \frac{1}{\lambda^2}, \quad \frac{\lambda_s}{\lambda} = A \end{cases}$$

which after time integration produces the global-in-time freezing regimes (1.8). The energy method is run along similar lines for very well localized initial data, since it now involves the confining measure $e^{\frac{By^2}{2}}dy$, $B > 0$. The analysis is slightly simpler thanks to the better decay of the B_k mode which induces a stronger decoupling from the remaining modes.

1.5. **Plan of the paper.** In section 2, we use a Lyapounov-Schmidt like argument to compute the bound state of $\mathcal{H}_b$ and the associated spectral gap in weighted norms in both the melting regime $\frac{\lambda_s}{\lambda} < 0$, Lemma 2.6, and the freezing regime $\frac{\lambda_s}{\lambda} > 0$, Lemma 2.8. In section 3, we construct the melting regimes. We introduce the nonlinear decomposition of the flow, section 3.1, compute the modulation equations using the free boundary geometry, sections 3.2 and 3.3, and close the key energy

[8]This is reminiscent of similar essential issues in [32].

[9]whereas the energy method in [42, 41, 33] works in both the dissipative and dispersive settings, but barely uses the dissipative terms in the energy estimates.

bound, Proposition 3.2. The proof of Theorem 1.1 now follows from a classical shooting argument à la Brouwer detailed in section 3.6. In section 4, we *deliberately* follow a parallel plan for the construction of the global-in-time freezing regimes.

2. **Spectral analysis in weighted spaces**

We compute in this section the k first eigenvalues of the linear operator

$$\mathcal{H}_{b,\pm} = -\Delta \mp b\Lambda \quad \text{with boundary condition} \quad u(1) = 0$$

and the associated spectral gap estimate in the perturbative regime $0 < b < b^*(k)$. The proof relies on a Lyapounov Schmidt type bifurcation argument at $b = 0$ performed in weighted Sobolev spaces.

To ease notations, we fix

$$\pm = -, \quad \mathcal{H}_b = \mathcal{H}_{b,-}, \quad \rho = \rho_- = e^{-\frac{|z|^2}{2}}, \quad b > 0, \quad \text{in sections 2.2, 2.3, 2.4} \tag{2.1}$$

and we omit the $-$ subscript for the sake of simplicity. The case $b < 0$ with the ρ_+ weight and the operator $\mathcal{H}_{b,+}$ is addressed in section 2.5.

2.1. **Coercivity for the harmonic oscillator.** We recall in this section without proof the classical estimates for the harmonic oscillator.

Melting case: consider $-\Delta + \Lambda$ on $(H^1_{\rho_-,0}, \langle\cdot,\cdot\rangle_0)$. This operator is self adjoint for the $\langle\cdot,\cdot\rangle_{0,-}$ scalar product as is easily seen by writing

$$-\Delta + \Lambda = -\frac{1}{\rho_- z}\partial_z\left(\rho_- z \partial_z\right). \tag{2.2}$$

The normalized Laguerre polynomials [45]

$$L_k(x) = \frac{e^x}{k!}\frac{d^k}{dx^k}(e^{-x}x^k), \quad k \in \mathbb{N}, \tag{2.3}$$

solve

$$XL_k'' + (1-X)L_k' + kL_k = 0$$

and hence

$$P_k(r) = L_k\left(\frac{r^2}{2}\right) \tag{2.4}$$

diagonalizes the harmonic oscillator:

$$(-\Delta + \Lambda)P_k = 2kP_k, \quad \langle P_n, P_m\rangle_{\rho,0} = 1. \tag{2.5}$$

Moreover, they satisfy the double normalization condition:

$$\int_0^{+\infty} L_n L_m e^{-x}dx = \delta_{nm}, \quad L_n(0) = 1$$

or equivalently

$$\langle P_j, P_k\rangle_{0,-} = \delta_{jk}, \quad P_k(0) = 1, \tag{2.6}$$

and the classical induction formula

$$\Lambda P_k = 2k(P_k - P_{k-1}), \quad k \geq 1. \tag{2.7}$$

An extensive overview of Laguerre polynomials can be found in [45]. We recall the standard sharp Poincaré inequality for the harmonic oscillator: $\forall u \in H^1_{\rho_-,0}$ with

$$\langle u, P_j\rangle_{0,-} = 0, \quad 0 \leq j \leq k,$$

there holds:

$$\|\partial_z u\|_{L^2_{\rho_-,0}} \geq (2k+2)\|u\|^2_{L^2_{\rho_-,0}}. \tag{2.8}$$

Freezing case: Consider $-\Delta - \Lambda$ on $(H^1_{\rho_+,0}, \langle\cdot,\cdot\rangle_0)$. Then the map

$$\begin{aligned} L^2_{0,+} &\to L^2_{0,-} \\ v &\mapsto w = e^{\frac{|z|^2}{2}} v \end{aligned}$$

is an isometry and integrating by parts:

$$\int |\nabla v|^2 e^{\frac{|z|^2}{2}} \rho_+ dz = \int |\nabla w|^2 \rho_- dz + 2B \int |w|^2 \rho_{B,-} dz \tag{2.9}$$

or equivalently:

$$(-\Delta - \Lambda)v = (-\Delta w + \Lambda w + 2w)\, e^{-\frac{|z|^2}{2}}. \tag{2.10}$$

Hence the family of eigenvectors

$$\hat{P}_j = P_j e^{-\frac{|z|^2}{2}}$$

diagonalizes the operator, and there holds the spectral gap estimate: $\forall u \in H^1_{\rho_+,0}$ with

$$\langle u, \hat{P}_j\rangle_{0,+} = 0, \quad 0 \leq j \leq k,$$

there holds:

$$\|\partial_z u\|_{L^2_{\rho_+,0}} \geq (2k+4)\|u\|^2_{L^2_{\rho_+,0}}. \tag{2.11}$$

Remark 2.1. The shift $2 = d$ in (2.11), where d stands for the dimension of the ambient space, will be crucial for the computation of the freezing rates.

2.2. **Almost invertibility of the renormalized operator.** Recall the notational convention (2.1). We consider the renormalized operator

$$H_b = -\Delta + \Lambda \ \text{ with boundary condition } \ u(\sqrt{b}) = 0$$

in the radial sector and for $0 < b < b^*$ small enough. Thanks to the boundary condition $u(\sqrt{b}) = 0$ and (2.2), H_b is self adjoint for the scalar product $\langle\cdot,\cdot\rangle_b$ on the Hilbert space $H^1_{\rho,b}$ given by (1.11). We claim a near invertibility property of H_b which is the starting point of the Lyapounov Schmidt argument.

Before stating the lemma, we introduce some notations. We first fix a frequency size

$$K \in \mathbb{N}$$

and a sufficiently small parameter

$$0 < b < b^*(K) \ll 1.$$

Universal constants in the sequel may depend on K, but are uniform in $b \in (0, b^*(K))$. We define the Gramm matrix

$$M_{b,k} = (\langle P_i, P_j\rangle_b)_{0 \leq i,j \leq k}, \quad k \leq K. \tag{2.12}$$

Observe from (2.5) that

$$\langle P_i, P_j\rangle_b = \langle P_i, P_j\rangle_0 + O(\int_{|z| \leq \sqrt{b}} z dz) = \delta_{i,j} + O(b) \tag{2.13}$$

and hence

$$M_{b,k} = Id + O(b) \ \text{ is invertible} \tag{2.14}$$

for $0 \leq b < b^*(k)$ small enough. We introduce the vector:

$$\mathcal{P}_k = (P_j)_{0 \leq j \leq k}$$

and consider the function

$$m_k(b,z) = (M_{b,k}^{-1}\mathcal{P}_k(\sqrt{b}), \mathcal{P}_k(z)), \tag{2.15}$$

which by (2.3) and (2.14) satisfies:

$$m_k(b,z) = \sum_{j=0}^{k} [1+O(b)]\, P_j(z). \tag{2.16}$$

We now claim:

Lemma 2.2 (Near inversion of $H_b - 2k$). *Let $k \in \mathbb{N}$ and $0 < b < b^*(k)$ small enough. Then for all $f \in L^2_{\rho,b}$ with*

$$\langle f, P_j\rangle_b = 0, \quad 0 \le j \le k, \tag{2.17}$$

there is a unique solution $u \in H^1_{\rho,b}$ to:

$$\left\{\begin{array}{l} \tilde{H}_{b,k}u = f \quad \textit{where} \quad \tilde{H}_{b,k}u = (H_b - 2k)u - \sqrt{b}m_k(b,z)\partial_z u(\sqrt{b}) \\ \langle u, P_j\rangle_b = 0, \quad 0 \le j \le k. \end{array}\right. \tag{2.18}$$

Moreover,

$$\|\Delta u\|_{L^2_{\rho,b}} + \|\partial_z u\|_{L^2_{\rho,b}} + \|\Lambda u\|_{L^2_{\rho,b}} + \|u\|_{L^2_{\rho,b}} + |\log b|\left|\sqrt{b}\partial_z u(\sqrt{b})\right| \lesssim \|f\|_{L^2_{\rho,b}}. \tag{2.19}$$

Proof of Lemma 2.2. We use a Lax Milgram type argument in $H^1_{\rho,b}$. Let $k \in \mathbb{N}$ and define the constraint set

$$\mathcal{C} := \{u \in H^1_{\rho,b}\,\big|\langle u, P_j\rangle_b = 0, \quad 0 \le j \le k\}.$$

We consider the problem of minimizing the functional $\mathcal{F} : H^1_{\rho,b} \to \mathbb{R}$:

$$\mathcal{F}(u) = \int_{z\ge\sqrt{b}} |\partial_z u|^2 \rho z dz - 2k\int_{z\ge\sqrt{b}} u^2\rho z dz - \langle f, u\rangle_b$$

over the constraint set $u \in \mathcal{C}$. Let

$$I_b = \inf_{u\in\mathcal{C}} \mathcal{F}(u).$$

We recall from the standard Poincaré inequality for the harmonic oscillator (2.8) and the compactness estimate (A.1) that for spherically symmetric $v \in H^1_{\rho,0}$ with $\langle v, P_j\rangle_0 = 0$, $0 \le j \le k$, there holds:

$$\int |\partial_z v|^2\rho z dz - 2k\int v^2\rho z dz \gtrsim \int (1+|z|^2)|v|^2\rho z dz. \tag{2.20}$$

Applying this to $v(z) = u\mathbf{1}_{z\ge\sqrt{b}} \in H^1_{\rho,0}$, we conclude that $I_b > -\infty$ and that any minimizing sequence u_n is uniformly bounded in $H^1_{\rho,b}$. Therefore, up to a subsequence, using the compact Sobolev embedding $H^1_{\rho,b} \hookrightarrow L^2_{\rho,b}$ and (2.20), we conclude:

$$u_n \rightharpoonup u \ \text{ in } \ H^1_{\rho,b}, \quad u_n \to u \ \text{ in } \ L^2_{\rho,b}.$$

In particular, using the local compactness of the embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, this implies

$$u(\sqrt{b}) = 0, \quad \langle u, P_j\rangle_b = 0, \quad 0 \le j \le k$$

and u is a minimizer of $\mathcal{F}$ over $\mathcal{C}$. By a standard variational argument, there exist Lagrange multipliers $\lambda_j \in \mathbb{R}$ such that

$$H_b u = f - \sum_{j=0}^{k} \lambda_j P_j. \tag{2.21}$$

Hence from standard regularity argument, $u \in H^2_{\rm loc}(r \geq \sqrt{b})$. We may then take the scalar product with P_i and compute:

$$\begin{aligned}\langle H_b u, P_i\rangle_b &= -\int_{z\geq\sqrt{b}} \frac{1}{z\rho}\partial_z(z\rho\partial_z u)P_i\rho z dz = \sqrt{b}e^{-b/2}P_j(\sqrt{b})\partial_z u(\sqrt{b}) + \langle u, H_b P_i\rangle_b\\ &= \sqrt{b}e^{-b/2}P_i(\sqrt{b})\partial_z u(\sqrt{b}) + 2i\langle u, P_i\rangle_b = \sqrt{b}e^{-b/2}P_i(\sqrt{b})\partial_z u(\sqrt{b}).\end{aligned}$$

We conclude from (2.21), (2.17), (2.12) that

$$\sqrt{b}e^{-b/2}\partial_z u(\sqrt{b})\left(P_i(\sqrt{b})\right)_{0\leq i\leq k} = -M_{b,k}(\lambda_i)_{0\leq i\leq k}$$

or equivalently

$$(\lambda_i)_{0\leq i\leq k} = -\sqrt{b}e^{-b/2}\partial_z u(\sqrt{b})M_{b,k}^{-1}\mathcal{P}_k(\sqrt{b})$$

and hence u solves (2.18) from the definition (2.15) and (2.21). We now observe that $u \in \mathcal{C}$ ensures

$$\langle m_k(b,\cdot), u\rangle_b = 0$$

and hence taking the scalar product of (2.18) with u, using (2.20) with $v = u\mathbf{1}_{z\geq\sqrt{b}}$, and the identity $\langle H_b u, u\rangle_b = \|\partial_z u\|^2_{L^2_{\rho,b}}$ yields:

$$\|u\|^2_{L^2_{\rho,b}} \lesssim \|\partial_z u\|^2_{L^2_{\rho,b}} - 2k\|u\|^2_{L^2_{\rho,b}} = \langle f, u\rangle_b$$

and hence

$$\|u\|_{L^2_{\rho,b}} + \|\partial_z u\|_{L^2_{\rho,b}} \lesssim \|f\|_{L^2_{\rho,b}}. \tag{2.22}$$

We now integrate by parts to compute:

$$\langle H_b u, \log z\rangle_b = \langle u, 1\rangle_b - \frac{1}{2}|\log b|\sqrt{b}e^{-b/2}\partial_z u(\sqrt{b}). \tag{2.23}$$

We estimate from (2.16)

$$\|m_k(b,\cdot)\|_{L^2_{\rho,b}} \lesssim 1 \tag{2.24}$$

and hence (2.23), (2.18) ensure

$$\left|\sqrt{b}\partial_z u(\sqrt{b})\right| \lesssim \frac{1}{|\log b|}\left(\|H_b u\|_{L^2_{\rho,b}} + \|u\|_{L^2_{\rho,b}}\right) \lesssim \frac{1}{|\log b|}\left(\|f\|_{L^2_{\rho,b}} + \left|\sqrt{b}\partial_z u(\sqrt{b})\right|\right)$$

which together with (2.22) yields

$$\|\partial_z u\|_{L^2_{\rho,b}} + \|u\|_{L^2_{\rho,b}} + |\log b|\left|\sqrt{b}\partial_z u(\sqrt{b})\right| \lesssim \|f\|_{L^2_{\rho,b}}.$$

We now use the equation (2.18) again and teh bound (2.24) which yield

$$\|H_b u\|_{L^2} \lesssim \|f\|_{L^2_{\rho,b}}.$$

We then use a Pohozaev type integration by parts to compute:

$$\begin{aligned}\int_{z\geq\sqrt{b}} z\partial_z u(H_b u)z\rho dz &= -\int_{z\geq\sqrt{b}}\partial_z(z\rho\partial_z u)z\rho\partial_z u\frac{dz}{\rho}\\ &= -\left[\frac{1}{2}z^2\rho^2(\partial_z u)^2\frac{1}{\rho}\right]_{\sqrt{b}}^{+\infty} - \frac{1}{2}\int_{z\geq\sqrt{b}} z^2\rho^2(\partial_z u)^2\frac{\partial_z\rho}{\rho^2}dz\\ &= \frac{1}{2}be^{-b/2}|\partial_z u(\sqrt{b})|^2 + \frac{1}{2}\int_{z\geq\sqrt{b}}(\Lambda u)^2 z\rho dz\\ &= O\left(\|f\|^2_{L^2_{\rho,b}}\right) + \frac{1}{2}\int_{z\geq\sqrt{b}}(\Lambda u)^2 z\rho dz.\end{aligned}$$

which after a simple application of the Cauchy-Schwarz inequality yields $\|\Lambda u\|_{L^2_{\rho,b}} \lesssim \|f\|_{L^2_{\rho,b}}$ and (2.19) is proven. □

2.3. **Partial diagonalization of H_b.** We are now in position to diagonalize H_b for frequencies $0 \le k \le K$ under the smallness $0 < b < b^*(K)$.

Proposition 2.3 (Eigenvalues for H_b)**.** *Let $K \in \mathbb{N}$. Then for all $0 < b < b^*(K)$ small enough, H_b admits a sequence of eigenvalues*

$$H_b\psi_{b,k} = \lambda_{b,k}\psi_{b,k}, \quad \psi_{b,k} \in H^1_{\rho,b}, \quad 0 \le k \le K, \tag{2.25}$$

such that for each $0 \le k \le K$, the following properties hold:
(i) Expansion of eigenvalues*: there holds the expansion of the eigenvalue*

$$\lambda_{b,k} = 2k + \frac{2}{|\mathrm{log} b|} + \tilde{\lambda}_{b,k}, \quad \tilde{\lambda}_{b,k} = O\left(\frac{1}{|\mathrm{log} b|^2}\right), \quad |\partial_b \lambda_{b,k}| \lesssim \frac{1}{b|\mathrm{log} b|^2}. \tag{2.26}$$

(ii) Expansion of eigenvectors*: there holds the expansion*

$$\left\{\begin{array}{l} \psi_{b,k} = T_{b,k}(z) + \tilde{\psi}_{b,k}(z) \\ T_{b,k}(z) = P_k(z)\left[\mathrm{log} z - \mathrm{log}(\sqrt{b})\right] + \sum_{j=0}^{k-1}\mu_{b,jk}P_j(z)\left[\mathrm{log} z - \mathrm{log}(\sqrt{b})\right] \end{array}\right. \tag{2.27}$$

with

$$\mu_{b,jk} = \frac{2}{(k-j)|\mathrm{log} b|} + \tilde{\mu}_{b,jk}, \quad \tilde{\mu}_{b,jk} = O\left(\frac{1}{|\mathrm{log} b|^2}\right), \quad b\partial_b\tilde{\mu}_{b,jk} = O\left(\frac{1}{|\mathrm{log} b|^3}\right) \tag{2.28}$$

and

$$\begin{aligned} &\|\Delta\tilde{\psi}_{b,k}\|_{L^2_{\rho,b}} + \|z^2\tilde{\psi}_{b,k}\|_{L^2_{\rho,b}} + \|\tilde{\psi}_{b,k}\|_{H^1_{\rho,b}} + \|\Lambda\tilde{\psi}_{b,k}\|_{L^2_{\rho,b}} + \|b\partial_b\tilde{\psi}_{b,k}\|_{L^2_{\rho,b}} \\ &\quad + |\mathrm{log} b|\left|\sqrt{b}\partial_z\tilde{\psi}_{b,k}(\sqrt{b})\right| \lesssim \frac{1}{|\mathrm{log} b|}, \end{aligned} \tag{2.29}$$

$$\|\partial_b\psi_{b,k}\|_{H^1_{\rho,b}} + |\mathrm{log} b||\sqrt{b}\partial_b\partial_z\psi_{b,k}(\sqrt{b})| \lesssim \frac{1}{b}, \tag{2.30}$$

and

$$\|\Lambda\psi_{b,k}\|_{L^2_{\rho,b}} + \|z^2\psi_{b,k}\|_{L^2_{\rho,b}} \lesssim |\mathrm{log} b|. \tag{2.31}$$

(iii) Spectral gap estimate*: let $u \in H^1_{\rho,b}$ with*

$$\langle u, \psi_{b,j}\rangle_b = 0, \quad 0 \le j \le k,$$

then

$$\|\partial_z u\|^2_{L^2_{\rho,b}} \ge \left[2k + 2 + O\left(\frac{1}{|\mathrm{log} b|}\right)\right]\|u\|^2_{L^2_{\rho,b}}. \tag{2.32}$$

Moreover,

$$\lambda_{b,0} = \inf_{u \in H^1_{\rho,b}} \frac{\|\partial_z u\|^2_{L^2_{\rho,b}}}{\|u\|^2_{L^2_{\rho,b}}} \tag{2.33}$$

and

$$\psi_{b,0}(z) > 0 \quad \textit{for} \quad z > \sqrt{b}. \tag{2.34}$$

(iv) Further identities*: there hold the algebraic identities for $0 \le j \le k$:*

$$\langle \psi_{b,k}, \psi_{b,k}\rangle_b = \frac{|\mathrm{log} b|^2}{4}\left[1 + O\left(\frac{1}{|\mathrm{log} b|}\right)\right]. \tag{2.35}$$

$$\frac{\langle b\partial_b\psi_{b,j}, \psi_{b,k}\rangle_b}{\langle \psi_{b,k}, \psi_{b,k}\rangle_b} = -\frac{1}{|\mathrm{log} b|}\delta_{jk} + O\left(\frac{1}{|\mathrm{log} b|^2}\right), \tag{2.36}$$

Moreover,

$$\|\Lambda\psi_{b,k} - 2k(\psi_{b,k} - \psi_{b,k-1})\|_{H^2_{\rho,b}} \lesssim 1, \tag{2.37}$$

$$\|b\partial_b\psi_{b,k} + \frac{1}{|\log b|}\psi_{b,k}\|_{H^2_{\rho,b}} \lesssim \frac{1}{|\log b|}, \tag{2.38}$$

$$\|z^2\psi_{b,k} + (2k+2)\psi_{b,k+1} - (4k+2)\psi_{b,k} + 2k\psi_{b,k-1}\|_{H^2_{\rho,b}} \lesssim 1. \tag{2.39}$$

Remark 2.4. From standard Sturm Liouville oscillation argument, $\psi_{b,k}$ vanishes k times on $z > \sqrt{b}$, and hence only the ground state $\psi_{b,0}$ is nonnegative.

Remark 2.5. Constants in Lemma 2.3 depend on the frequency K but are uniform in $0 < b < b^*(K)$.

Proof of Proposition 2.3. The argument is of Lyapounov-Schmidt type. We remove the b subscript as much as possible to simplify notations.

step 1 The Lyapunov-Schmidt argument. Let $T_k(z) \in H^1_{\rho,b}$ be given by (2.27), and introduce the universal profile

$$\mathcal{T}_b(z) := \log z - \log(\sqrt{b}).$$

Then

$$\left\{\begin{array}{l} (H_b - 2k)T_k(z) = Q_k(z) + \sum_{j=0}^{k-1}\mu_{jk}\left[2(j-k)\mathcal{T}_b(z)P_j(z) + Q_j\right] \\ Q_j = -2\frac{P_j'(z)}{z} + P_j(z). \end{array}\right. \tag{2.40}$$

Observe from (2.4) that $Q_j = \tilde{Q}_j(r^2)$ with $\deg \tilde{Q}_j \le j-1$ and hence

$$\forall 0 \le j \le k, \quad Q_j \in \mathrm{Span}(P_j)_{0\le j\le k}. \tag{2.41}$$

We solve the eigenvalue problem

$$(H_b - 2k)\psi = \mu_k\psi, \tag{2.42}$$

by representing ψ in the form

$$\psi = T_k + \tilde{\psi}. \tag{2.43}$$

and hence (2.42), (2.43) give an equation for $\tilde{\psi}$:

$$\begin{aligned} (H_b - 2k)\tilde{\psi} &= -(H_b - 2k)T_k + \mu_k(T_k + \tilde{\psi}) \\ &= -Q_k(z) - \sum_{j=0}^{k-1}\mu_{jk}\left[2(j-k)\mathcal{T}_b(z)P_j(z) + Q_j\right] + \mu_k(T_k + \tilde{\psi}) \\ &=: f(\tilde{\psi}). \end{aligned} \tag{2.44}$$

We define $(\mu_j(\tilde{\psi}))_{0\le j\le k}$ by imposing the relations:

$$\frac{\langle f(\tilde{\psi}), P_i\rangle_b}{\langle P_i, P_i\rangle_b} = \sqrt{b}\partial_z\tilde{\psi}(\sqrt{b})\left(M^{-1}_{b,k}\mathcal{P}_k(\sqrt{b})\right)_i, \quad i = 0, \dots, k, \tag{2.45}$$

which is proved below to correspond to an invertible linear system on $((\mu_{j,k})_{0\le j\le k-1}, \mu_k)$. Observe that (2.45) allows us to rewrite (2.44) as:

$$\tilde{H}_{b,k}\tilde{\psi} = f - \sum_{j=0}^{k}\frac{\langle f, P_j\rangle_b}{\|P_j\|^2_{L^2_{\rho,b}}}P_j = F(\tilde{\psi}) \tag{2.46}$$

with $\tilde{H}_{b,k}$ given by (2.18). Thus to find $\tilde{\psi}$, by Lemma 2.2 we are left with solving the fixed point equation

$$\tilde{\psi} = \tilde{H}^{-1}_{b,k}F(\tilde{\psi}).$$

and we indeed claim that the operator $\tilde{H}_{b,k}^{-1} \circ F$ is a strict contraction from the closed ball

$$B_\alpha := \Big\{ \tilde{\psi} \in H^1_{\rho,b} \Big| \; \|\Delta\tilde{\psi}\|_{L^2_{\rho,b}} + \sqrt{b}|\partial_z \tilde{\psi}(\sqrt{b})| + \|\tilde{\psi}\|_{H^1_{\rho,b}} \le \frac{\alpha}{|\log b|}, \\ \tilde{\psi}(\sqrt{b}) = 0, \quad \langle \tilde{\psi}, P_j \rangle_b = 0, \quad 0 \le j \le k \Big\}, \tag{2.47}$$

to itself for α universal large enough.

Computation of μ_{jk}, μ_k: We invert (2.45). Indeed, we rewrite

$$\begin{aligned} f(\tilde{\psi}) &= -Q_k + \frac{|\log b|}{2}\mu_k P_k \left[1 + \frac{2\log z}{|\log b|}\right] + \mu_k \tilde{\psi} \\ &- \sum_{j=0}^{k-1} \frac{|\log b|}{2} \mu_{jk} \left\{ [2(j-k) - \mu_k] P_j \left[1 + \frac{2\log z}{|\log b|}\right] + \frac{2Q_j}{|\log b|} \right\}. \end{aligned}$$

Let

$$\vec{\mu} = (\mu_{0k}, \dots, \mu_{(k-1)k}, \mu_k) \quad \text{and} \quad |\vec{\mu}|_\infty := \max\{|\mu_{jk}|_{0\le j\le k-1}, |\mu_k|\}.$$

Using the almost orthogonality (2.13) and the $\langle \cdot, \cdot \rangle_b$-orthogonality of $\tilde{\psi}$ and P_j, $j = 0, \dots k$, we obtain:

$$\frac{\langle f(\tilde{\psi}), P_j \rangle_b}{\langle P_j, P_j \rangle_b} = (k-j)|\log b|\mu_{jk} - \langle Q_k, P_j \rangle_0 + (C\vec{\mu})_j + \mu_k \sum_{\ell=0}^{k-1} \mu_{\ell k} \langle P_j, P_\ell \mathcal{T}_b \rangle_b + O(|b|), \tag{2.48}$$

$$\frac{\langle f(\tilde{\psi}), P_k \rangle_b}{\langle P_k, P_k \rangle_b} = \mu_k \frac{|\log b|}{2} - \langle Q_k, P_k \rangle_0 + \mu_k \sum_{\ell=0}^{k-1} \mu_{\ell k} \langle P_k, P_\ell \mathcal{T}_b \rangle_b + (C\vec{\mu})_k + O(|b|), \tag{2.49}$$

where $C = (C_{ij})_{i,j=0,\dots k} = O(1)$, is a $(k+1) \times (k+1)$-matrix with bounded entries in the regime where b is small and $(C\vec{\mu})_i = \sum_{\ell=0}^{k-1} C_{i\ell}\mu_{\ell k} + C_{ik}\mu_k$ is the i-th entry of the vector $C\vec{\mu}$, $i = 0, \dots, k$. The above system contains quadratic terms and it can be solved for $\vec{\mu}$ by the following simple iteration argument. Given $\vec{\tilde{\mu}} = (\tilde{\mu}_{0k}, \dots . \tilde{\mu}_{(k-1)k}, \tilde{\mu}_k)$, consider the system

$$\begin{aligned} \frac{\langle f(\tilde{\psi}), P_j \rangle_b}{\langle P_j, P_j \rangle_b} &= (k-j)|\log b|\mu_{j,k} - \langle Q_k, P_j \rangle_0 + (\tilde{C}\vec{\mu})_j + O(|b|), \\ \frac{\langle f(\tilde{\psi}), P_k \rangle_b}{\langle P_k, P_k \rangle_b} &= \mu_k \frac{|\log b|}{2} - \langle Q_k, P_k \rangle_0 + (\tilde{C}\vec{\mu})_k + O(|b|), \end{aligned}$$

where $\tilde{C}_{ij} = C_{ij} + \delta_{ik} \sum_{\ell=0}^{k-1} \tilde{\mu}_{ik} \langle P_j, P_\ell \mathcal{T}_b \rangle_b$, $j = 0, \dots k-1$. Assuming that $|\vec{\tilde{\mu}}|_\infty \le 1$, we have $\tilde{C} = O(1)$ and we can invert the above system to obtain

$$\mu_{jk} = \frac{\langle Q_k, P_j \rangle_0}{(k-j)|\log b|} + O\left(\frac{1}{|\log b|^2} + \frac{\sqrt{b}|\partial_z \tilde{\psi}(\sqrt{b})|}{|\log b|} \right), 0 \le j \le k-1, \tag{2.50}$$

$$\mu_k = \frac{2\langle Q_k, P_k \rangle_0}{|\log b|} + O\left(\frac{1}{|\log b|^2} + \frac{\sqrt{b}|\partial_z \tilde{\psi}(\sqrt{b})|}{|\log b|} \right) \tag{2.51}$$

and therefore $|\vec{\mu}|_\infty \lesssim \frac{1}{|\log b|} \le 1$ for $b < b^*$ sufficienty small, where we used (2.47). Contractive property follows easily in a similar manner and for a given $\tilde{\psi} \in B_\alpha$ we

obtain the unique solution $\vec{\mu}$. We may moreover integrate by parts and use (2.6) to compute:

$$\langle Q_k, P_k\rangle_0 = \langle -2\frac{P_k'}{z} + P_k, P_k\rangle_0 = P_k^2(0) = 1.$$

Similarily for $j \le k-1$:

$$\langle Q_k, P_j\rangle_0 = -2\langle \frac{P_k'}{z}, P_j\rangle_0 = 2 + \langle \frac{P_j'}{z}, P_k\rangle_0 = 2$$

since $\frac{P_j'}{z}$ is a polynomial of r^2 of degree $\le j-1 < k$. Note that by (2.50) and (2.51) we additionally have the bound

$$|\vec{\mu}|_\infty \lesssim \frac{1}{|\log b|} + \frac{\sqrt{b}|\partial_z\tilde{\psi}(\sqrt{b})|}{|\log b|}. \tag{2.52}$$

<u>*Estimating $F(\tilde{\psi})$*</u>: Let us introduce the approximate projection operator:

$$\mathbb{P}_k f := \sum_{j=0}^{k} \frac{\langle f, P_j\rangle_b}{\|P_j\|^2_{L^2_{\rho,b}}} P_j.$$

Observe that $(\mathrm{I}-\mathbb{P}_k)P_j = O(b)$ and by (2.41) $(\mathrm{I}-\mathbb{P}_k)Q_j = O(b)$ for any $j \in \{0, 1, \dots, k\}$ and the almost orthogonality relation (2.13). From (2.47) we have that $(\mathrm{I}-\mathbb{P}_k)\tilde{\psi} = \tilde{\psi}$. Thus, from (2.44) and (2.46) we obtain that

$$\begin{aligned}
F(\tilde{\psi}) &= \mu_k(\mathrm{I}-\mathbb{P}_k)\left[P_k\log z + P_k\frac{|\log b|}{2}\right] + \mu_k\tilde{\psi} \\
&\quad + \sum_{j=0}^{k-1}\mu_{j,k}\left[2(k-j)+\mu_k\right](\mathrm{I}-\mathbb{P}_k)\left[P_j\log z + P_j\frac{|\log b|}{2}\right] + O(b(1+|\vec{\mu}|_\infty)) \\
&= \mu_k(\mathrm{I}-\mathbb{P}_k)\left[P_k\log z\right] + \mu_k\tilde{\psi} + \sum_{j=0}^{k-1}\mu_{j,k}\left[2(k-j)+\mu_k\right](\mathrm{I}-\mathbb{P}_k)\left[P_j\log z\right] \\
&\quad + O\left(b(1+|\vec{\mu}|_\infty) + b|\log b||\vec{\mu}|_\infty\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|F(\tilde{\psi})\|_{L^2_{\rho,b}} &\lesssim |\vec{\mu}|_\infty\left(1 + \|\tilde{\psi}\|_{L^2_{\rho,b}}\right) + O\left(b(1+|\vec{\mu}|_\infty) + b|\log b||\vec{\mu}|_\infty\right) \\
&\le C\frac{1+\sqrt{b}|\partial_z\tilde{\psi}(\sqrt{b})|}{|\log b|}\left(1 + \frac{\alpha}{|\log b|}\right),
\end{aligned} \tag{2.53}$$

for $0 < b < b^*$ with b^* sufficiently small and a universal constant $C > 0$. Applying Lemma 2.2 and using (2.53) we conclude from (2.46)

$$\begin{aligned}
&\|\Delta\tilde{\psi}\|_{L^2_{\rho,b}} + \|\tilde{\psi}\|_{H^1_{\rho,b}} + |\log b||\sqrt{b}\partial_z\tilde{\psi}(\sqrt{b})| \lesssim \|F(\tilde{\psi})\|_{L^2_{\rho,b}} \\
&\le C\frac{1+\sqrt{b}|\partial_z\tilde{\psi}(\sqrt{b})|}{|\log b|}\left(1+\frac{\alpha}{|\log b|}\right)
\end{aligned}$$

and hence

$$\|\Delta\tilde{\psi}\|_{L^2_{\rho,b}} + \|\tilde{\psi}\|_{H^1_{\rho,b}} + |\log b||\sqrt{b}\partial_z\tilde{\psi}(\sqrt{b})| + \|F(\tilde{\psi})\|_{L^2_{\rho,b}} \le \frac{\alpha}{|\log b|}$$

and $\tilde{\psi} \in B_\alpha$ for $\alpha > 0$ universal large enough and $0 < b < b^*(k)$ small enough. Therefore $\tilde{\psi} \in B_\alpha$.

<u>*Contractive property*</u>: To show the contractive property, note that for any $\phi_1, \phi_2 \in B_\alpha$ by (2.44) we have that

$$
\begin{aligned}
& f(\phi_1) - f(\phi_2) \\
&= \frac{|\log b|}{2}\left(\mu_k(\phi_1) - \mu_k(\phi_2)\right) P_k \left[1 + \frac{2\log z}{|\log b|}\right] + \left(\mu_k(\phi_1) - \mu_k(\phi_2)\right)\phi_1 + \mu_k(\phi_2)\left(\phi_1 - \phi_2\right) \\
&\quad - \sum_{j=0}^{k-1} \frac{|\log b|}{2}\left(\mu_{jk}(\phi_1) - \mu_{jk}(\phi_2)\right)\left\{\left[2(j-k) - \mu_k\right] P_j \left[1 + \frac{2\log z}{|\log b|}\right] + \frac{2Q_j}{|\log b|}\right\}.
\end{aligned}
$$

By a calculation analogous to (2.48) and (2.49), we can evaluate the $\langle \cdot, \cdot \rangle_b$-inner product of $f(\phi_1) - f(\phi_2)$ with P_j, $j = 0, 1, \dots, k$ and thereby estimate $|\mu_{jk}(\phi_1) - \mu_{jk}(\phi_2)|$, $|\mu_k(\phi_1) - \mu_k(\phi_2)|$. Using (2.47), we arrive at

$$
\|f(\phi_1) - f(\phi_2)\|_{L^2_{\rho,b}} \lesssim \frac{1}{|\log b|}\|\phi_1 - \phi_2\|_{L^2_{\rho,b}},
$$

which together with $F = (\mathrm{Id} - \mathbb{P}_k)f$ gives $\|F(\phi_1) - F(\phi_2)\|_{L^2_{\rho,b}} \lesssim \frac{1}{|\log b|}\|\phi_1 - \phi_2\|_{L^2_{\rho,b}}$. The operator $\tilde{H}_b^{-1}$ is continuous by Lemma 2.2 and therefore for a sufficiently small $0 < b < b_*$ the operator $\tilde{H}_b^{-1} \circ F$ is a strict contraction. By the Banach fixed point theorem, there exists a unique $\psi_{b,k} \in H^1_{\rho,b}$ satisfying (2.25). The Fréchet differentiability of ψ_b with respect to b at any fixed $b > 0$ follows similarly, the classical details are left to the reader.

step 2 Proof of (2.31). We estimate from (2.28), (2.27):

$$
\|\Lambda T_{b,k}\|_{L^2_{\rho,b}} \lesssim |\log b|
$$

and (2.29) now implies

$$
\|\Lambda \psi_{b,k}\|_{L^2_{\rho,b}} \lesssim |\log b|. \tag{2.54}
$$

We may now apply (A.1) to $z\psi_b$ and (2.31) follows from (2.54).

step 3 Spectral gap estimate. Let $u \in H^1_{\rho,b}$ satisfy

$$
\langle u, \psi_{b,j}\rangle_b = 0, \quad 0 \le j \le k.
$$

Let

$$
v = u\mathbf{1}_{z \ge \sqrt{b}} - \sum_{j=0}^{k} \frac{\langle u, P_j\rangle_b}{\|P_j\|_{L^2_{\rho,0}}} P_j \in H^1_{\rho,0}
$$

then by the Poincaré inequality (2.8):

$$
\|\partial_z v\|^2_{L^2_{\rho,b}} \ge (2k+2)\|v\|^2_{L^2_{\rho,b}}. \tag{2.55}
$$

We now compute from (2.27) for $0 \le j \le k$:

$$
0 = \langle u, \psi_j\rangle_b = \left\langle u, P_j(z)\mathcal{T}_b(z) + \sum_{i=0}^{j-1} \mu_i \left[P_i(z) - P_i(\sqrt{b})\right] + \tilde{\psi}_{b,k} \right\rangle_b
$$

and hence using (2.28), (2.29):

$$
|\langle u, P_j\rangle_b| \lesssim \frac{1}{|\log b|}\|u\|_{L^2_{\rho,b}}, \quad 0 \le j \le k
$$

from which

$$
\|u - v\|_{H^1_{\rho,b}} \lesssim \frac{\|u\|_{L^2_{\rho,b}}}{|\log b|}.
$$

Injecting this into (2.55) implies

$$\|\partial_z u\|^2_{L^2_{\rho,b}} \geq \left[2k+2+O\left(\frac{1}{|\log b|}\right)\right]\|u\|^2_{L^2_{\rho,b}}. \tag{2.56}$$

To prove (2.33), (2.34), let

$$\mu_b = \inf_{u\in\mathcal{C}_b}\frac{\|\partial_z u\|^2_{L^2_{\rho,b}}}{\|u\|^2_{L^2_{\rho,b}}}.$$

Then any minimizing sequence normalized by $\|u_n\|_{L^2_{\rho,b}} = 1$ is bounded in $H^1_{\mathrm{loc}}(r \geq \sqrt{b})$. By the compactness of radial Sobolev and trace embeddings and (A.1) it strongly converges in $L^2_{\rho,b}$. Hence the infimum is attained and from Lagrange multiplier theory:

$$H_b\phi_b = \mu_b\phi_b.$$

Moreover, since $|\phi_b|$ is also an infimum, we may assume $\phi_b \geq 0$. If $\mu_b \neq \lambda_b$, then $\langle\phi_b,\psi_{b,0}\rangle_b = 0$ and hence from (2.32):

$$\mu_b = \|\partial_z\phi_b\|^2_{L^2_{\rho,b}} \gtrsim 1.$$

Note that $\lambda_{b,0} = \frac{\|\partial_z\psi_b\|^2_{L^2_{\rho,b}}}{\|\psi_b\|^2_{L^2_{\rho,b}}} \geq \mu_b$ by the definition of μ_b. Together with (2.26) this contradicts the definition of μ_b for $0 < b < b^*$ small enough. Hence $\lambda_{b,0} = \mu_b$ is the bound state. The simplicity of the first eigenvalue follows from a classical argument [6]- and hence $\psi_b \equiv \phi_b \geq 0$. Note that $\psi_b > 0$ for $z > \sqrt{b}$ by the strong maximum principle.

step 4 Estimate for $\partial_b\lambda_{b,k}$. Applying ∂_b to (2.25), we obtain

$$H_b\partial_b\psi_{b,k} = \partial_b\lambda_{b,k}\psi_{b,k} + \lambda_{b,k}\partial_b\psi_{b,k}. \tag{2.57}$$

Evaluating the $\langle\cdot,\cdot\rangle_b$ inner product of (2.57) with $\psi_{b,k}$ and integrating by parts we obtain

$$\begin{aligned}\partial_b\lambda_{b,k}\|\psi_{b,k}\|^2_{L^2_{\rho,b}} + \lambda_{b,k}\langle\partial_b\psi_{b,k},\psi_{b,k}\rangle_b &= \langle H_b\partial_b\psi_{b,k},\psi_{b,k}\rangle_b\\ &= \langle\partial_b\psi_{b,k}, H_b\psi_{b,k}\rangle_b - \partial_b\psi_{b,k}(\sqrt{b})\partial_z\psi_{b,k}(\sqrt{b})\rho(\sqrt{b})\sqrt{b}\\ &= \lambda_{b,k}\langle\partial_b\psi_{b,k},\psi_{b,k}\rangle_b - \partial_b\psi_{b,k}(\sqrt{b})\partial_z\psi_{b,k}(\sqrt{b})\rho(\sqrt{b})\sqrt{b}.\end{aligned}$$

Therefore

$$\partial_b\lambda_{b,k} = -\frac{\partial_b\psi_{b,k}(\sqrt{b})\partial_z\psi_{b,k}(\sqrt{b})\rho(\sqrt{b})\sqrt{b}}{\|\psi_{b,k}\|^2_{L^2_{\rho,b}}}. \tag{2.58}$$

From $\psi_{b,k}(\sqrt{b}) = 0$ it follows that $\partial_b\psi_{b,k}(\sqrt{b}) = -\frac{1}{2\sqrt{b}}\partial_z\psi_{b,k}(\sqrt{b})$ and therefore from (2.58)

$$\partial_b\lambda_{b,k} = \frac{|\partial_z\psi_{b,k}(\sqrt{b})|^2\rho(\sqrt{b})}{2\|\psi_{b,k}\|^2_{L^2_{\rho,b}}}. \tag{2.59}$$

In particular, since $|\partial_z\psi_{b,k}(\sqrt{b})| = O(\frac{1}{\sqrt{b}})$ by (2.27) and (2.29), and $\|\psi_{b,k}\|^2_{L^2_{\rho,b}} \gtrsim |\log b|^2$ by (2.28) - (2.29) it follows that

$$|\partial_b\lambda_{b,k}| \lesssim \frac{1}{b|\log b|^2}. \tag{2.60}$$

which is the last claim of (2.26).

step 5 Estimate for $|\partial_b \mu_{b,ik}|$, $i = 0, \dots, k-1$. From (2.27) we obtain

$$\partial_b T_{b,k} = -\frac{1}{2b} P_k + \sum_{j=0}^{k-1} \partial_b \mu_{b,jk} P_j \mathcal{T}_b - \frac{1}{2b} \sum_{j=0}^{k-1} \mu_{b,jk} P_j. \tag{2.61}$$

From (2.27) and (2.57) it follows that

$$H_b \partial_b \tilde{\psi}_{b,k} = F_k + \lambda_{b,k} \partial_b \tilde{\psi}_{b,k}, \tag{2.62}$$

where

$$F_k = -H_b \partial_b T_{b,k} + \partial_b (\lambda_{b,k} T_{b,k}) + \partial_b \lambda_{b,k} \tilde{\psi}_{b,k}. \tag{2.63}$$

Rewriting (2.62) in the form $H_b \partial_b \psi_{b,k} = (\partial_b \lambda_{b,k} \tilde{\psi}_{b,k} + \lambda_{b,k} \partial_b \tilde{\psi}_{b,k}) + \lambda_{b,k} \partial_b T_{b,k}$ and evaluating the $\langle \cdot, \cdot \rangle_b$-inner product with P_j, $j = 0, \dots, k-1$, we obtain

$$\langle H_b \partial_b \psi_{b,k}, P_j \rangle_b = \lambda_{b,k} \langle \partial_b T_{b,k}, P_j \rangle_b \tag{2.64}$$

since $\langle \tilde{\psi}_{b,k}, P_j \rangle_b = \langle \partial_b \tilde{\psi}_{b,k}, P_j \rangle_b = 0$. On the other hand, from (2.61) we have that

$$\begin{aligned} \langle \partial_b T_{b,k}, P_j \rangle_b &= -\frac{1}{2b} M_{jk} + \sum_{i=0}^{k-1} \partial_b \mu_{b,ik} \left(M_{ji} \frac{|\log b|}{2} + \langle P_i \log z, P_j \rangle_b \right) - \frac{1}{2b} \sum_{i=0}^{k-1} \mu_{b,ik} M_{ji} \\ &= \frac{|\log b|}{2} \partial_b \mu_{b,ik} \delta_{ji} + \sum_{i=0}^{k-1} c_{ji} \partial_b \mu_{b,ik} + O(\frac{1}{b|\log b|}), \end{aligned} \tag{2.65}$$

where $(c_{ij})_{i,j=0,\dots,k-1} = O(1)$ and we used $M_{ji} = \delta_{ji} + O(b)$ for $j = 0, \dots, k-1$ and the first two claims of (2.28) which have already been proven above. On the other hand, observe that by integration-by-parts and the orthogonality $\langle \partial_b \tilde{\psi}_{b,k}, P_j \rangle_b = 0$ we have

$$\langle H_b \partial_b T_{b,k}, P_j \rangle_b = \rho(\sqrt{b}) \sqrt{b} \left(\partial_z \partial_b T_{b,k}(\sqrt{b}) P_j(\sqrt{b}) - \partial_b T_{b,k}(\sqrt{b}) \partial_z P_j(\sqrt{b}) \right). \tag{2.66}$$

By (2.61) $|\partial_b T_{b,k}(\sqrt{b})| \lesssim 1/b$ and

$$\partial_z \partial_b T_{b,k} = -\frac{1}{2b} \left(\partial_z P_k(\sqrt{b}) + \sum_{j=0}^{k-1} \mu_{b,jk} \partial_z P_j(\sqrt{b}) \right) + \frac{1}{\sqrt{b}} \sum_{j=0}^{k-1} \partial_b \mu_{b,jk} P_j(\sqrt{b}),$$

which together with (2.66) leads to

$$\langle H_b \partial_b T_{b,k}, P_j \rangle_b = \rho(\sqrt{b}) \sum_{j=0}^{k-1} \partial_b \mu_{b,jk} P_j(\sqrt{b}) + O(1), \tag{2.67}$$

where we note that $\partial_z P_k(\sqrt{b}) = \sqrt{b} L_k'(\frac{b}{2}) = O(\sqrt{b})$ implying $\frac{1}{\sqrt{b}} \max_{j=0,\dots,k} |\partial_z P_k(\sqrt{b})| = O(1)$. To see that $L_k'(\frac{b}{2}) = O(1)$ observe that by the definition (2.3) of the k-th Laguerre polynomial, it follows that L_k' is a polynomial of degree $k-1$. From (2.64), (2.65), and (2.67) we conclude that

$$\frac{|\log b|}{2} \partial_b \mu_{b,ik} \delta_{ji} + c_{ji}^* \partial_b \mu_{b,ik} = O(\frac{1}{b|\log b|}), \quad c_{ji}^* = O(1), \;\; i,j = 0, \dots, k-1. \tag{2.68}$$

The system (2.68) is invertible for $0 \le b \le b^*$ sufficiently small, and as a consequence

$$\sup_{i=0,\dots,k-1} |\partial_b \mu_{b,ik}| \lesssim \frac{1}{b|\log b|^2}, \tag{2.69}$$

which completes the proof of (2.28).

step 5 Estimate for $\|\partial_b\tilde\psi_{b,k}\|_b$. Recalling that $\langle\tilde\psi_{b,k},P_j\rangle_b=0$, $j=0,1,\dots k$, by the construction of $\tilde\psi_{b,k}$, we conclude that $\langle\partial_b\tilde\psi_{b,k},P_j\rangle_b=0$ since $\tilde\psi_{b,k}(\sqrt{b})=0$. Moreover, the spectral gap estimate (2.8) with $u=\partial_b\tilde\psi_{b,k}\mathbf{1}_{z\geq\sqrt{b}}+\partial_b\tilde\psi_{b,k}(\sqrt{b})\mathbf{1}_{0\leq z<\sqrt{b}}$ together with the bound $\lambda_{b,k}=2k+\tilde\lambda_{b,k}$ imply

$$\lambda_{b,k}\|\partial_b\psi_{b,k}\|^2_{L^2_{\rho,b}}\lesssim\frac{2k+O(\frac{1}{|\log b|})}{2k+2+O(\frac{1}{|\log b|})}\|\partial_z\partial_b\psi_{b,k}\|^2_{L^2_{\rho,b}}+C|\partial_b\tilde\psi_{b,k}(\sqrt{b})|^2. \tag{2.70}$$

Evaluating the $\langle\cdot,\cdot\rangle_b$ inner product of (2.62) with $\partial_b\tilde\psi_{b,k}$, integrating-by-parts, using Cauchy-Schwarz, and the Poincaré-type inequality (2.70) we obtain

$$\|\partial_z\partial_b\tilde\psi_{b,k}\|^2_{L^2_{\rho,b}}\lesssim\|F_k\|^2_{L^2_{\rho,b}}+\sqrt{b}|\partial_z\partial_b\tilde\psi_{b,k}(\sqrt{b})\partial_b\tilde\psi_{b,k}(\sqrt{b})|+|\partial_b\tilde\psi_{b,k}(\sqrt{b})|^2. \tag{2.71}$$

From $\tilde\psi_{b,k}(\sqrt{b})=0$ it follows after differentiating with respect to b that

$$|\partial_b\tilde\psi_{b,k}(\sqrt{b})|\lesssim\frac{1}{\sqrt{b}}|\partial_z\tilde\psi_{b,k}(\sqrt{b})|\lesssim\frac{1}{b|\log b|^2}. \tag{2.72}$$

To estimate $\sqrt{b}|\partial_z\partial_b\tilde\psi_{b,k}(\sqrt{b})|$ we note that

$$\begin{aligned}0&=\langle\partial_b\tilde\psi_{b,k},1\rangle_b=\langle\partial_b\tilde\psi_{b,k},H_b\log z\rangle_b\\&=\langle H_b\partial_b\tilde\psi_{b,k},\log z\rangle_b+e^{-b/2}\partial_b\tilde\psi_{b,k}(\sqrt{b})+\frac12|\log b|\sqrt{b}\partial_z\partial_b\tilde\psi_{b,k}(\sqrt{b})e^{-b/2}.\end{aligned}$$

Therefore

$$\begin{aligned}\left|\sqrt{b}\partial_z\partial_b\tilde\psi_{b,k}(\sqrt{b})\right|&\lesssim\frac{1}{|\log b|}\left(\|F_k\|_{L^2_{\rho,b}}+|\lambda_{b,k}|\|\partial_b\tilde\psi_{b,k}\|_{L^2_{\rho,b}}+\frac{1}{b|\log b|^2}\right)\\&\lesssim\frac{1}{|\log b|}\|F_k\|_{L^2_{\rho,b}}+\frac{1}{|\log b|}\|\partial_z\partial_b\tilde\psi_{b,k}\|_{L^2_{\rho,b}}+\frac{1}{b|\log b|^3},\end{aligned}$$

where we used (2.62) in the first estimate and the Poincaré inequality (2.70) in the last. Using the Cauchy-Schwarz inequality and plugging this bound and (2.72) into (2.71) we obtain

$$\|\partial_z\partial_b\tilde\psi_{b,k}\|_{L^2_{\rho,b}}\lesssim\|F_k\|_{L^2_{\rho,b}}+\frac{1}{b|\log b|^2}. \tag{2.73}$$

From (2.61) and (2.69) we conclude that $\|T_{b,k}\|_{L^2_{\rho,b}}\lesssim\frac1b$. To estimate $\|F_k\|_{L^2_{\rho,b}}$ note that from (2.40) we have the identity:

$$H_b\partial_bT_{b,k}=2k\partial_bT_{b,k}+\sum_{j=0}^{k-1}\partial_b\mu_{b,jk}\left[2(j-k)\mathcal{T}_bP_j+Q_j\right]+\frac1b\sum_{j=0}^{k-1}\mu_{b,jk}(k-j)P_j$$

and therefore using (2.61) and (2.26) we have

$$\begin{aligned}-H_b\partial_bT_{b,k}+\lambda_{b,k}\partial_bT_{b,k}&=O(\frac{1}{|\log b|})\partial_bT_{b,k}\\&\quad+\sum_{j=0}^{k-1}\partial_b\mu_{b,jk}\left[2(j-k)\mathcal{T}_bP_j+Q_j\right]+\frac1b\sum_{j=0}^{k-1}\mu_{b,jk}(k-j)P_j.\end{aligned}$$

Hence, using (2.69) and (2.28):

$$\|-H_b\partial_bT_{b,k}+\lambda_{b,k}\partial_bT_{b,k}\|_{L^2_{\rho,b}}\lesssim\frac{1}{b|\log b|}.$$

Thus, using the definition (2.63) of F_k, bounds (2.60), (2.29), and the previous bound we obtain

$$\|F_k\|_{L^2_{\rho,b}} \lesssim \frac{1}{b|\log b|} + |\partial_b \lambda_{b,k}| \left(\|T_{b,k}\|_{L^2_{\rho,b}} + \|\tilde{\psi}_{b,k}\|_{L^2_{\rho,b}} \right)$$
$$\lesssim \frac{1}{b} + \frac{1}{b|\log b|^2} \left(|\log b| + \frac{1}{|\log b|} \right) \lesssim \frac{1}{b|\log b|}.$$

Plugging this back into (2.73) we get

$$\|\partial_z \partial_b \tilde{\psi}_{b,k}\|_{L^2_{\rho,b}} \lesssim \frac{1}{b|\log b|}$$

and therefore, by the spectral gap estimate (2.8), just like in (2.70), we obtain

$$\|\partial_b \tilde{\psi}_{k,b}\|_{L^2_{\rho,b}} \lesssim \frac{1}{b|\log b|}$$

and the proof of (2.29) is completed.

step 6 Proof of (2.35) - (2.39).
Proof of (2.35). We estimate from (2.27), (2.29):

$$\langle \psi_{b,k}, \psi_{b,k} \rangle_b = \frac{|\log b|^2}{4} \left[\langle P_k, P_k \rangle_0 + O\left(\frac{1}{|\log b|} \right) \right] \tag{2.74}$$

and (2.35) follows from the normalization (2.6).
Proof of (2.36). We compute from (2.61), (2.28), (2.29):

$$\left\| b\partial_b \psi_{b,k} + \frac{1}{2} P_k \right\|_{H^2_{\rho,b}} \lesssim \frac{1}{|\log b|} \tag{2.75}$$

and hence using (2.35):

$$\begin{aligned} \frac{\langle b\partial_b \psi_{b,j}, \psi_{b,k} \rangle_b}{\langle \psi_{b,k}, \psi_{b,k} \rangle_b} &= \frac{4}{(\log b)^2} \left[1 + O\left(\frac{1}{\log b} \right) \right] \left[\langle -\frac{P_k}{2}, \psi_{b,j} \rangle_b + O(1) \right] \\ &= \frac{4}{(\log b)^2} \left[-\frac{1}{4} |\log b| \langle P_j, P_k \rangle_b + O(1) \right] = -\frac{1}{|\log b|} + O\left(\frac{1}{|\log b|^2} \right), \end{aligned}$$

this is (2.36).
Proof of (2.37): We compute from (2.27), (2.7):

$$\begin{aligned} \Lambda \psi_{b,k} &= \Lambda P_k \mathcal{T}_b + P_k + \sum_{j=0}^{k-1} \mu_{b,jk} \Lambda (P_j \mathcal{T}_b) + \Lambda \tilde{\psi}_{b,k} \\ &= 2k[P_k - P_{k-1}] \mathcal{T}_b + P_k + \sum_{j=0}^{k-1} \mu_{b,jk} \Lambda (P_j \mathcal{T}_b) + \Lambda \tilde{\psi}_{b,k} \\ &= 2k(\psi_{b,k} - \psi_{b,k-1}) + \mathcal{E}_{b,k} \end{aligned}$$

where the remainder estimate

$$\|\mathcal{E}_{b,k}\|_{H^2_{\rho,b}} \lesssim 1$$

holds due to (2.28), (2.29) and (2.37) is proved.
Proof of (2.38). Note that

$$P_k = \frac{2}{|\log b|} \left\{ \psi_{b,k} - \tilde{\psi}_{b,k} - P_k \log z - \sum_{i=0}^{k-1} \mu_{b,jk} P_j \mathcal{T}_b \right\}$$

and therefore

$$\begin{aligned}
&\|b\partial_b\psi_{b,k}+\frac{1}{|\mathrm{log} b|}\psi_{b,k}\|_{H^2_{\rho,b}}\\
\lesssim\ &\|b\partial_b\psi_{b,k}+\frac{1}{2}P_k\|_{H^2_{\rho,b}}+\frac{1}{|\mathrm{log} b|}\|\tilde{\psi}_{b,k}+P_k\mathrm{log} z+\sum_{i=0}^{k-1}\mu_{b,jk}P_j\mathcal{T}_b\|_{H^2_{\rho,b}}\\
\lesssim\ &\frac{1}{|\mathrm{log} b|},
\end{aligned}$$

where we used (2.75) and (2.28), (2.29). This concludes the proof of (2.38).
Proof of (2.39). We have from (2.27), (2.29):

$$\|z^2\psi_{b,k}-\frac{|\log b|}{2}z^2P_k\|_{H^1_b}\lesssim 1.$$

Let $\Phi_k=z^2P_k$, then

$$(-\Delta+\Lambda)\Phi_k=(2k+2)\Phi_k-4P_k-4\Lambda P_k=(2k+2)\Phi_k-4P_k-8k(P_k-P_{k-1})$$

and hence the relation

$$\begin{aligned}
&2k\langle\Phi_k,P_k\rangle_0=(2k+2)\langle\Phi_k,P_k\rangle_0-4-8k \quad \text{ie} \quad \langle\Phi_k,P_k\rangle_0=2+4k\\
&(2k-2)\langle\Phi_j,P_{k-1}\rangle_0=(2k+2)\langle\Phi_K,P_{k-1}\rangle_0+8k \quad \text{ie} \quad \langle\Phi_k,P_k\rangle_0=-2k\\
&\langle\Phi_k,P_j\rangle_0=0, \quad 0\le j\le k-2.
\end{aligned}$$

Since P_k is a polynomial we conclude that there exists a $c_k\in\mathbb{R}$ such that

$$z^2P_k=c_kP_{k+1}+(4k+2)P_k-2kP_{k-1}.$$

Since $P_k(0)=1$, we obtain by plugging in $z=0$ into the above relationship:

$$z^2P_k=-2(k+1)P_{k+1}+(4k+2)P_k-2kP_{k-1},$$

which yields (2.39). □

2.4. **Diagonalization of $\mathcal{H}_b$.** We are now position to derive the bound state and the spectral gap estimate for the operator $\mathcal{H}_b$.

Lemma 2.6 (Renormalized eigenfunction)**.** *Let $K\in\mathbb{N}$. Then for all $0<b<b^*(K)$ small enough, the renormalized operator*

$$\mathcal{H}_b=-\Delta+b\Lambda \quad \textit{with boundary value} \quad u(1)=0$$

has a family of eigenstates $\eta_{b,k}$ satisfying:

$$\mathcal{H}_b\eta_{b,k}=b\lambda_{b,k}\eta_{b,k}, \quad 0\le k\le K, \tag{2.76}$$

with the following properties:
(i) Structure of the eigenmodes*: there holds the expansion*

$$\left\{\begin{array}{l}\eta_{b,k}=S_{b,k}(y)+\tilde{\eta}_{b,k}(y)\\ S_{b,k}(y)=P_k(\sqrt{b}y)\mathrm{log} y+\sum_{j=0}^{k-1}\mu_{b,jk}P_j(\sqrt{b}y)\mathrm{log} y\end{array}\right. \tag{2.77}$$

with

$$\frac{\|\Delta\tilde{\eta}_{b,k}\|_{L^2_b}}{\sqrt{b}}+\|\partial_y\tilde{\eta}_{b,k}\|_b+\sqrt{b}\|\tilde{\eta}_{b,k}\|_b+\sqrt{b}\|\Lambda\tilde{\eta}_{b,k}\|_b+|\mathrm{log} b||\partial_y\tilde{\eta}_{b,k}(1)|\lesssim\frac{1}{|\mathrm{log} b|}, \tag{2.78}$$

(ii) Further estimates on the eigenvector*: there holds*

$$\|y\eta_{b,k}\|_b \lesssim \frac{|\mathrm{log} b|}{b}, \quad \|y^2\eta_{b,k}\|_b \lesssim \frac{|\mathrm{log} b|}{b\sqrt{b}} \tag{2.79}$$

$$\|b\partial_b\eta_{b,k}\|_b \lesssim \frac{|\mathrm{log} b|}{\sqrt{b}}. \tag{2.80}$$

Moreover:

$$\|\Lambda\eta_{b,k} - 2k(\eta_{b,k} - \eta_{b,k-1})\|_b + \frac{1}{\sqrt{b}}\|\partial_y[\Lambda\eta_{b,k} - 2k(\eta_{b,k} - \eta_{b,k-1})]\|_b$$
$$+\frac{1}{b}\|\mathcal{H}_b\left[\Lambda\eta_{b,k} - 2k(\eta_{b,k} - \eta_{b,k-1})\right]\|_b \lesssim \frac{1}{\sqrt{b}} \tag{2.81}$$

and

$$\|2b\partial_b\eta_{b,j} - \Lambda\eta_{b,j} + \frac{2}{|\mathrm{log} b|}\eta_{b,j}\|_b + \frac{1}{\sqrt{b}}\|\partial_y[2b\partial_b\eta_{b,j} - \Lambda\eta_{b,j} + \frac{2}{|\mathrm{log} b|}\eta_{b,j}\|_b$$
$$+\frac{1}{b}\|\mathcal{H}_b\left[2b\partial_b\eta_{b,j} - \Lambda\eta_{b,j} + \frac{2}{|\mathrm{log} b|}\eta_{b,j}\right]\|_b \lesssim \frac{1}{\sqrt{b}|\mathrm{log} b|}. \tag{2.82}$$

(iii) Normalization:

$$(\eta_{b,k}, \eta_{b,k})_b = \frac{|\mathrm{log} b|^2}{4b}\left[1 + O\left(\frac{1}{|\mathrm{log} b|}\right)\right]. \tag{2.83}$$

Remark 2.7. Observe that (2.81), (2.82), (2.83) imply the bound:

$$\|b\partial_b\eta_{b,k} - k(\eta_{b,k} - \eta_{b,k-1})\|_b + \frac{1}{\sqrt{b}}\|\partial_y[b\partial_b\eta_{b,k} - k(\eta_{b,k} - \eta_{b,k-1})]\|_b$$
$$+\frac{1}{b}\|\mathcal{H}_b\left[b\partial_b\eta_{b,k} - k(\eta_{b,k} - \eta_{b,k-1})\right]\|_b \lesssim \frac{1}{\sqrt{b}}. \tag{2.84}$$

Proof of Lemma 2.6. Given $u : \Omega \to \mathbb{R}$, let $v(y) = u(\sqrt{b}y)$, it is straightforward to check that

$$\mathcal{H}_b v = b(H_b u)(\sqrt{b}y), \quad \|\partial_y^\ell v\|_b = b^{\frac{\ell-1}{2}}\|\partial_z^\ell u\|_{L^2_{\rho,b}}, \quad \ell \in \mathbb{N}. \tag{2.85}$$

We therefore let

$$\eta_{b,k}(y) = \psi_{b,k}(z), \quad z = y\sqrt{b}. \tag{2.86}$$

and (2.76) follows. The estimate (2.81) follows by rescaling (2.37), and (2.83) by rescaling (2.35). The decomposition (2.77) follows from (2.27), and (2.29) implies (2.78). The bounds (2.79) follow by rescaling the bounds:

$$\|z\psi_{b,k}\|_{L^2_{\rho,b}} + \|z^2\psi_{b,k}\|_{L^2_{\rho,b}} \lesssim |\mathrm{log} b|.$$

Directly from the definition (2.86), we compute:

$$b\partial_b\eta_{b,k} = \left[\frac{\Lambda\psi_{b,k}}{2} + b\partial_b\psi_{b,k}\right](\sqrt{b}y), \tag{2.87}$$

which together with (2.30), (2.31) yields:

$$\|b\partial_b\eta_{b,k}\|_b \lesssim \frac{|\mathrm{log} b|}{\sqrt{b}},$$

thus proving (2.80). From (2.87):

$$2b\partial_b\eta_{b,k} - \Lambda\eta_{b,k} = 2\left[b\partial_b\psi_{b,k}\right](\sqrt{b}y)$$

and hence from (2.38):

$$\left\|2b\partial_b\eta_{b,k}-\Lambda\eta_{b,k}+\frac{2}{|\log b|}\eta_{b,k}\right\|_b\lesssim\frac{1}{\sqrt{b}}\|b\partial_b\psi_{b,k}+\frac{1}{|\log b|}\psi_{b,k}\|_{L^2_{\rho_b}}\lesssim\frac{1}{|\log b|\sqrt{b}}$$

and similarly for higher derivatives. □

2.5. **Diagonalization of $\mathcal{H}_B$.** We now change sign and consider the operator for $B>0$

$$\mathcal{H}_B=-\Delta-B\Lambda\ \text{ with boundary value }\ u(1)=0, \tag{2.88}$$

which is a self adjoint operator on $H^1_{B,+}$ given by (1.12).

Lemma 2.8 (Renormalized eigenfunction)**.** *Let $K\in\mathbb{N}$. Then for all $0<B<B^*(K)$ small enough, the renormalized operator*

$$\mathcal{H}_B=-\Delta-B\Lambda\ \text{ with boundary value }\ u(1)=0$$

has a family of eigenstates

$$\hat{\eta}_{B,k}=e^{-\frac{B|y|^2}{2}}\eta_{B,k},\ \ \mathcal{H}_B\hat{\eta}_{B,k}=B\hat{\lambda}_{B,K}\hat{\eta}_{B,k},\ \ \hat{\lambda}_{B,k}=\lambda_{B,k}+2,\quad 0\le k\le K, \tag{2.89}$$

with $\eta_{B,k},\lambda_{B,k}$ given by Lemma 2.6. Furthermore, there hold the following properties:
(i) Structure of the eigenmodes*: there holds the expansion*

$$\left\{\begin{array}{l}\hat{\eta}_{B,k}=S_{B,k}(y)e^{-\frac{B|y|^2}{2}}+\tilde{\hat{\eta}}\\ S_{B,k}(y)=P_k(\sqrt{B}y)\log y+\sum_{j=0}^{k-1}\mu_{B,jk}P_j(\sqrt{B}y)\log y\end{array}\right. \tag{2.90}$$

with

$$\frac{\|\Delta\tilde{\hat{\eta}}_{B,k}\|_{L^2_B}}{\sqrt{B}}+\|\partial_y\tilde{\hat{\eta}}_{B,k}\|_B+\sqrt{B}\|\tilde{\hat{\eta}}_{B,k}\|_B+\sqrt{B}\|\Lambda\tilde{\hat{\eta}}_{B,k}\|_B+|\log B||\partial_y\tilde{\hat{\eta}}_{B,k}(1)|$$
$$\lesssim\frac{1}{|\log B|}, \tag{2.91}$$

(ii) Further estimates on the eigenvector*: there holds*

$$\|y\hat{\eta}_{B,k}\|_b\lesssim\frac{|\log B|}{B},\quad\|y^2\hat{\eta}_{B,k}\|_B\lesssim\frac{|\log B|}{B\sqrt{B}} \tag{2.92}$$

$$\|B\partial_B\hat{\eta}_{B,k}\|_b\lesssim\frac{|\log B|}{\sqrt{B}}. \tag{2.93}$$

Moreover:

$$\begin{aligned}&\|B\partial_B\hat{\eta}_{B,k}-(k+1)[\hat{\eta}_{B,k+1}-\hat{\eta}_{B,k}]\|_B\\ +\ &\frac{1}{\sqrt{B}}\|\partial_y[B\partial_B\hat{\eta}_{B,k}-(k+1)[\hat{\eta}_{B,k+1}-\hat{\eta}_{B,k}]\|_B\\ +\ &\frac{1}{B}\|\mathcal{H}_B\left[B\partial_B\hat{\eta}_{B,k}-(k+1)[\hat{\eta}_{B,k+1}-\hat{\eta}_{B,k}]\right]\|_B\lesssim\frac{1}{\sqrt{B}}\end{aligned} \tag{2.94}$$

and

$$\|2B\partial_B\hat{\eta}_{B,j}-\Lambda\hat{\eta}_{B,j}+\frac{2}{|\log B|}\hat{\eta}_{B,j}\|_b+\frac{1}{\sqrt{B}}\|\partial_y[2B\partial_B\hat{\eta}_{B,j}-\Lambda\hat{\eta}_{B,j}+\frac{2}{|\log B|}\hat{\eta}_{B,j}\|_B$$
$$+\frac{1}{B}\|\mathcal{H}_B\left[2B\partial_B\hat{\eta}_{B,j}-\Lambda\eta_{B,j}+\frac{2}{|\log B|}\hat{\eta}_{B,j}\right]\|_B\lesssim\frac{1}{\sqrt{B}|\log B|}. \tag{2.95}$$

(iii) Normalization:

$$(\hat{\eta}_{B,k},\hat{\eta}_{B,k})_B=\frac{|\log B|^2}{4B}\left[1+O\left(\frac{1}{|\log B|}\right)\right]. \tag{2.96}$$

Proof of Lemma 2.97. This is a direct consequence of Lemma 2.6. Indeed, the map

$$\begin{array}{l} L^2_{B,+}\to L^2_{B,-} \\ v\mapsto w=e^{\frac{B|y|^2}{2}}v \end{array} \quad \text{is an isometry} \tag{2.97}$$

and integrating by parts:

$$\int|\nabla v|^2e^{\frac{B|y|^2}{2}}\rho_{B,+}dy=\int|\nabla w|^2\rho_{B,-}dy+2B\int|w|^2\rho_{B,-}dy \tag{2.98}$$

or equivalently:

$$\mathcal{H}_Bv=(-\Delta w+B\Lambda w+2Bw)\,e^{-\frac{B|y|^2}{2}}.$$

Together with Lemma 2.6, this yields (2.89). We now renormalize (A.1) which yields:

$$B^{\frac{k+1}{2}}\|y^kw\|_{B,-}\lesssim\|\partial_yw\|_{B,-}+\sqrt{B}\|w\|_{B,-}. \tag{2.99}$$

The isometric relation (2.98) and (2.99) imply the following comparison of norms:

$$\|\partial_yv\|_{B,+}\lesssim\|\partial_yw\|_{B,-}+\sqrt{B}\|w\|_{B,+} \tag{2.100}$$

$$\|\Lambda v\|_{B,+}=\|\Lambda w-By^2w\|_{B,-}\lesssim\|\Lambda w\|_{B,-}+\frac{1}{\sqrt{B}}\|\partial_yw\|_{B,-}+\|w\|_{B,-} \tag{2.101}$$

$$\|y^kv\|_{B,+}\lesssim\|y^kw\|_{B,-}. \tag{2.102}$$

Using (2.99):

$$\begin{aligned}\|\Delta v\|_{B,+}&\lesssim\|\Delta w\|_{B,-}+B\|w\|_{B,-}+B\|\Lambda w\|_{B,-}+B^2\|y^2w\|_{B,-}\\&\lesssim\|\Delta w\|_{B,-}+B\|w\|_{B,-}+B\|\Lambda w\|_{B,-}+\sqrt{B}\|\partial_yw\|_{B,-}.\end{aligned} \tag{2.103}$$

We also observe that

$$|\partial_yw(1)|\lesssim|\partial_yv(1)|\quad\text{if}\quad w(1)=0.$$

Using the above bounds together with (2.89), (2.90), (2.78), (2.79), and (2.83) yields (2.91), (2.92), and (2.96). Moreover,

$$\partial_B\hat{\eta}_{B,k}=\left(\partial_B\eta_{B,k}-\frac{|y|^2}{2}\eta_{B,k}\right)e^{-\frac{By^2}{2}}.$$

Hence (2.93) follows from (2.80), (2.79). We now observe the fundamental conjugation

$$\Lambda\hat{\eta}_{B,j}-2B\partial_B\hat{\eta}_{B,j}=(\Lambda\eta_{B,j}-2B\partial_B\eta_{B,j})\,e^{-\frac{B|y|^2}{2}},$$

and hence (2.95) follows from (2.82). Moreover from (2.39):

$$B|y|^2\eta_{B,k}=z^2\psi_{B,k}=-(2k+2)\eta_{B,k+1}+(4k+2)\eta_{B,k}-2k\eta_{B,k-1}+F_k,\quad\|F_k\|_B\lesssim\frac{1}{\sqrt{B}}$$

and hence using (2.84):

$$\begin{aligned}B\partial_B\eta_{B,k}-B\frac{|y|^2}{2}\eta_{B,k}&=k(\eta_{B,k}-\eta_{B,k-1})-\frac{1}{2}\left[-(2k+2)\eta_{B,k+1}+(4k+2)\eta_{B,k}-2k\eta_{B,k-1}\right]+\tilde{F}_k\\&=(k+1)\left[\eta_{B,k+1}-\eta_{B,k}\right]+\tilde{F}_k\quad\text{with}\quad\|\tilde{F}_k\|_B\lesssim\frac{1}{\sqrt{B}}\end{aligned}$$

and similarily for higher derivatives, and (2.94) is proved. □

3. Finite time melting regimes

This section is devoted to the existence and stability of the melting process. In all the section, we let
$$\pm = -, \quad \rho = \rho_-, \quad b > 0.$$

3.1. **Renormalized equations and initialization.** We start with the classical modulated nonlinear decomposition of the flow. We let
$$u(t,x) = v(s,y), \quad y = \frac{r}{\lambda(t)}, \quad \lambda(0) = 1, \tag{3.1}$$
where $\lambda(0) = 1$ is assumed without loss of generality thanks to the scaling symmetry (1.14). We define the renormalized time
$$s(t) = s_0 + \int_0^t \frac{d\tau}{\lambda^2(\tau)}, \quad s_0 \gg 1, \tag{3.2}$$
and obtain the renormalized flow:
$$\left\{\begin{array}{l} \partial_s v + \mathcal{H}_a v = 0, \quad a = -\frac{\lambda_s}{\lambda}, \\ v(s,1) = 0, \quad \partial_y v(s,1) = a. \end{array}\right. \tag{3.3}$$
We now prepare our initial data in the following way:

case $k=0$: We first claim that given $0 < b^* \ll 1$ and ε^* with
$$\|\varepsilon^*\|_{b^*}^2 \lesssim \frac{b^*}{|\log b^*|^2},$$
there exists a locally unique decomposition
$$b^*\eta_{b^*,0} + \varepsilon^* = b\eta_{b,0} + \varepsilon \ \text{ with } \ (\varepsilon, \eta_{b,0})_b = 0, \quad |b - b^*| \lesssim \frac{b^*}{|\log b^*|^2}. \tag{3.4}$$
Indeed, we define the map
$$F(b, \varepsilon^*) = (b^*\eta_{b^*,0} - b\eta_{b,0} + \varepsilon^*, \eta_{b,0})_b$$
which satisfies $F(b^*, 0) = 0$ and
$$\partial_b F(b^*, 0) = -(\eta_{b^*,0} + b^*(\partial_b \eta_{b,0})|_{b=b^*}, \eta_{b^*,0})_{b^*} = -\frac{|\log b^*|^2}{4b^*}\left[1 + O\left(\frac{1}{|\log b^*|}\right)\right] < 0$$
from (2.83) and the degeneracy (2.84) for $k=0$. The claim then follows from a standard application of the implicit function theorem. A Taylor expansion of F about $(b, \varepsilon^*) \equiv (b^*, 0)$ yields the bound
$$|b - b^*| \frac{|\log b^*|^2}{4b^*}\left[1 + O\left(\frac{1}{|\log b^*|}\right)\right] \lesssim \|\varepsilon^*\|_{b^*}\|\eta_{b^*,0}\|_{b^*} \lesssim \|\varepsilon^*\|_{b^*}\frac{|\log b^*|}{2\sqrt{b^*}}$$
and hence
$$|b - b^*| \lesssim \frac{\sqrt{b^*}}{|\log b|^*}\frac{\sqrt{b^*}}{|\log b|^*}$$
which concludes the proof of (3.4).

We therefore pick an initial datum
$$v_0^* = b_0^*\eta_{b_0^*} + \varepsilon_0^*, \quad \|\varepsilon_0^*\|_{b^*}^2 \lesssim \frac{b^*}{|\log b^*|^2}$$
and decompose the solution
$$v(s,y) = b_0(s)\eta_{b_0(s),0} + \varepsilon(s,y) \ \text{ with } \ b(s) = b_0(s) \ \text{ and } \ (\varepsilon, \eta_{b,0})_b = 0 \tag{3.5}$$

which makes sense as long as $\varepsilon(s,y)$ is small enough in the L^2 weighted sense. We let

$$\varepsilon_2 = \mathcal{H}_b\varepsilon \tag{3.6}$$

and define the energy

$$\mathcal{E} := \|\mathcal{H}_b\varepsilon\|_b^2,$$

which is a coercive norm thanks to the orthogonality condition (3.5), see Appendix A. We assume the initial smallness

$$\mathcal{E}(0) \le \frac{b^3(0)}{|\mathrm{log} b(0)|^2}. \tag{3.7}$$

<u>case $k \ge 1$</u>: Let

$$c_{k,1} = -\frac{k+1}{2k^2}, \quad c_{k,2} = c_{k,1} - \frac{(k+1)\alpha_k}{k}. \tag{3.8}$$

Then we freeze the explicit value

$$b(s) := \frac{1}{2ks} + \frac{c_{k,1}}{s\mathrm{log}s} \tag{3.9}$$

and the sequence

$$\begin{cases} b_k^e = \frac{k+1}{2ks} + \frac{c_{k,2}}{s\mathrm{log}s}, \\ b_j^e = 0, \quad 0 \le j \le k-1, \end{cases} \tag{3.10}$$

where the index "e" stands for exact.

Remark 3.1. Letting

$$a^e = \frac{k+1}{2ks} + \frac{c_{k,1}}{s\mathrm{log}s}, \tag{3.11}$$

then (a^e, b, b_k^e) satisfy

$$\begin{cases} (b_k^e)_s + bb_k^e\left[2k + \frac{2}{|\log b|}\right] + \frac{2(a^e-b)b_k^e}{|\log b|} = O\left(\frac{1}{s^2(\mathrm{log}s)^2}\right) \\ b_s + 2b(a^e - b) = O\left(\frac{1}{s^2(\mathrm{log}s)^2}\right) \\ a^e - b_k^e\left(1 + \frac{2\alpha_k}{|\log b|}\right) = O\left(\frac{1}{s(\log s)^2}\right). \end{cases} \tag{3.12}$$

We define

$$Q_\beta(y) = \sum_{j=0}^{k} b_j \eta_{b,j}(y) \tag{3.13}$$

and introduce the dynamical decomposition

$$v(s,y) = Q_{\beta(s)} + \varepsilon(s,y), \quad (\eta_{b,j}(s), \varepsilon)_b = 0, \quad 0 \le j \le k. \tag{3.14}$$

We let again

$$\varepsilon_2 = \mathcal{H}_b\varepsilon, \quad \mathcal{E} := \|\mathcal{H}_b\varepsilon\|_b^2,$$

which due to the orthogonality conditions (3.14) is a coercive norm, see Appendix A. We assume the initial smallness

$$\mathcal{E}(0) \le \frac{b^3(0)}{|\mathrm{log} b(0)|}. \tag{3.15}$$

For $k \ge 1$, the set of initial data will be built as a codimension k manifold. To this end and in order to prepare the data, we consider the decomposition

$$b_j(s) = b_j^e(s) + \tilde{b}_j(s), \quad \tilde{b}_j(s) = \frac{V_j(s)}{s(\mathrm{log}s)^{\frac{3}{2}}}, \quad j = 0, \dots, k. \tag{3.16}$$

Let the (2×2)-matrix A_k be given by

$$A_k := \begin{pmatrix} -1 & -1 \\ 1 & 1+d_k \end{pmatrix}, \quad d_k := \frac{1}{k(k+1)}. \tag{3.17}$$

The matrix A_k is diagonalizable with one strictly positive $\mu_1^k > 0$ and one strictly negative eigenvalue $\mu_2^k < 0$. Let P_k be an orthogonal matrix diagonalizing A_k, i.e.

$$A_k = P_k^{-1}\Lambda_k P_k, \quad \Lambda_k := \begin{pmatrix} \mu_1^k & 0 \\ 0 & \mu_2^k. \end{pmatrix}$$

We define the new unknowns W_k, W_{k-1} by setting

$$\begin{pmatrix} W_k \\ W_{k-1} \end{pmatrix} := P_k \begin{pmatrix} V_k \\ V_{k-1} \end{pmatrix} \tag{3.18}$$

and now assume the initial bound

$$|W_k(0)| \le 1 \tag{3.19}$$

$$|W_{k-1}(0)|^2 + \sum_{j=0}^{k-2} \left|\frac{V_j(0)}{\delta}\right|^2 \le K^2 \tag{3.20}$$

for some universal constants $K > 0$, $0 < \delta(k) \ll 1$ to be chosen later. We then consider the bootstrap bounds

$$\mathcal{E} \le \begin{cases} \frac{Db^3}{|\log b|^2} & \text{for } k = 0, \\ \frac{Db^3}{|\log b|} & \text{for } k \ge 1 \end{cases} \tag{3.21}$$

for some large enough universal $D = D(k)$ to be chosen later, and

- for $k = 0$
$$0 < b_0(s) < b^* \tag{3.22}$$
- for $k \ge 1$,
$$|W_k(s)| \le K \tag{3.23}$$
and
$$|W_{k-1}(s)|^2 + \sum_{j=0}^{k-2} \left|\frac{V_j(s)}{\delta}\right|^2 \le K^2 \tag{3.24}$$

and define

$$s^* = \begin{cases} \sup_{s\ge s_0}\{(3.21), (3.22) \text{ hold on } [s_0, s]\} & \text{for } k = 0, \\ \sup_{s\ge s_0}\{(3.21), (3.23), (4.13) \text{ hold on } [s_0, s]\} & \text{for } k \ge 1. \end{cases}$$

The main ingredient of the proof of theorem 1.1 is the following:

Proposition 3.2 (Bootstrap estimates on b and ε)**.** *The following statements hold:*
1. Stable regime*: for $k = 0$, $s^* = +\infty$.*
2. Unstable regime*: for $k \ge 1$, there exist constants K, $\delta = \delta(K) \ll 1$ and $(V_0(0), \dots, V_{k-2}(0), W_{k-1}(0))$ depending on $\varepsilon(0)$ satisfying* (3.15)*,* (3.19) *and* (3.20)*, such that $s^* = +\infty$.*

Remark 3.3. Let us observe that our set of initial data is non empty and contains compactly supported arbitrarily small data in $\dot{H}^1$, see Appendix C.

Remark 3.4. The proof of the Proposition 3.2 is presented in section 3.5.

From now on and for the rest of this section, we study the flow in the boostrap regime $s \in [s_0, s^*)$. Note in particular the rough bounds

$$|b_k| \lesssim b, \quad |b_j| \lesssim \frac{b}{|\log b|}, \quad 0 \leq j \leq k-1 \quad \text{and} \quad \mathcal{E} \leq \frac{b^3}{\sqrt{|\log b|}} \tag{3.25}$$

for $s_0 \geq s_0(K)$ large enough.

3.2. **Extraction of the leading order ODE's driving the melting.** We derive in this section the main dynamical constraint on the parameters $(a, b, (b_j)_{0 \leq j \leq k})$ which lead to the leading order modulation equations, and are a combination of the linear diagonalization of the $\mathcal{H}_b$ operator and the nonlinear boundary conditions.

We start with the constraint induced by the boundary conditions.

Lemma 3.5 (Boundary conditions)**.** *There holds:*

$$a = \sum_{j=0}^{k} b_j \left[1 + \frac{2\alpha_j}{|\log b|}\right] + O\left(\frac{b}{|\log b|^2} + \frac{\sqrt{\mathcal{E}}}{|\log b|\sqrt{b}}\right), \tag{3.26}$$

$$\varepsilon_2(1) = -a(a-b), \tag{3.27}$$

$$\partial_y \varepsilon_2(1) = -a_s - \sum_{j=0}^{k} \lambda_{b,j} b b_j \left[1 + \frac{2\alpha_j}{|\log b|}\right] + O\left(\frac{b^2}{|\log b|^2}\right). \tag{3.28}$$

Proof of Lemma 3.5. We compute from (2.77), (2.78), (2.28) and recalling the definition (1.13):

$$\partial_y \eta_{b,j}(1) = 1 + \sum_{i=0}^{j-1} \frac{2}{(j-i)|\log b|} + O\left(\frac{1}{|\log b|^2}\right) = 1 + \frac{2\alpha_j}{|\log b|} + O\left(\frac{1}{|\log b|^2}\right). \tag{3.29}$$

This implies that

$$\partial_y Q_\beta(1) = \sum_{j=0}^{k} b_j \partial_y \eta_{b,j}(1) = \sum_{j=0}^{k} b_j \left[1 + \frac{2\alpha_j}{|\log b|}\right] + O\left(\frac{b}{|\log b|^2}\right).$$

Since $v = Q_\beta + \varepsilon$, it follows that

$$\begin{aligned} \varepsilon_y\big|_{y=1} &= v_y\big|_{y=1} - \partial_y Q_\beta\big|_{y=1} = -\frac{\lambda_s}{\lambda} - \partial_y Q_\beta(1) \\ &= a - \sum_{j=0}^{k} b_j \left[1 + \frac{2\alpha_j}{|\log b|}\right] + O\left(\frac{b}{|\log b|^2}\right), \end{aligned}$$

which together with (A.10) yields (3.26). From (3.3), $v(s,1) = 0$ and $\partial_y v(s,1) = -\frac{\lambda_s}{\lambda} = a$:

$$0 = \mathcal{H}_a v(1) = (\mathcal{H}_b v + (a-b)\Lambda v)(1) = \varepsilon_2(1) + a(a-b),$$

this is (3.27). Now from $\partial_y v(s,1) = a$, we have

$$\partial_s \partial_y v(s,1) = a_s.$$

On the other hand, taking ∂_y of (3.3), we have:

$$0 = \partial_s \partial_y v + \partial_y(\mathcal{H}_b v + (a-b)\Lambda v) = \partial_s \partial_y v + \partial_y \varepsilon_2 + \partial_y \mathcal{H}_b Q_\beta + (a-b) y \Delta v.$$

We evaluate the above identity at $y = 1$. From (3.3) and $\partial_s v(s,1) = 0$, $\partial_y v(s,1) = a$:

$$\Delta v(1) = a \Lambda v(1) = a^2.$$

By construction,

$$\partial_y \mathcal{H}_b Q_\beta = \sum_{j=0}^{k} \lambda_{b,j} b b_j \partial_y \eta_{b,j},$$

and hence:

$$a_s + \partial_y \varepsilon_2(1) + \sum_{j=0}^{k} \lambda_{b,j} b b_j \left[1 + \frac{2\alpha_j}{|\mathrm{log} b|}\right] + a^2(a-b) = O\left(\frac{b^2}{|\mathrm{log} b|^2}\right).$$

We inject into the estimate the rough bound

$$|a| \lesssim b \tag{3.30}$$

which follows from (3.25), (3.26) and (3.28) is proved. □

We now show how Q_β is prepared to generate an approximate solution to (3.3) with the suitable leading order dynamical system for (β, λ) induced by the spectral diagonalization of $\mathcal{H}_b$.

Proposition 3.6 (Leading order modulation equations)**.** *Under the a priori bounds of Proposition 3.2, there holds*

$$\partial_s Q_\beta + \mathcal{H}_a Q_\beta = \mathrm{Mod} + \Psi \tag{3.31}$$

where we defined the modulation vector

$$\begin{aligned} \mathrm{Mod} \quad := & \quad \left[(b_k)_s + b b_k \lambda_{b,k} + \frac{2(a-b)b_k}{|\mathrm{log} b|} + \frac{k b_k}{b}\Phi\right] \eta_{b,k} \\ + & \quad \sum_{j=0}^{k-1} \left[(b_j)_s + b b_j \lambda_{b,j} + \frac{2(a-b)b_j}{|\mathrm{log} b|} + \frac{j b_j - (j+1) b_{j+1}}{b}\Phi\right] \eta_{b,j}, \end{aligned} \tag{3.32}$$

the deviation

$$\Phi := b_s + 2b(a-b), \tag{3.33}$$

and the remaining error satisfies the bound:

$$\|\Psi\|_b + \frac{1}{\sqrt{b}}\|\partial_y \Psi\|_b + \frac{1}{b}\|\mathcal{H}_b \Psi\|_b \lesssim \frac{b^{\frac{3}{2}}}{|\mathrm{log} b|} + \frac{|\Phi|}{\sqrt{b}}. \tag{3.34}$$

Proof of Proposition 3.6. By definition

$$\mathcal{H}_a = \mathcal{H}_b + (a-b)\Lambda$$

and we therefore compute from (3.13):

$$\begin{aligned} \partial_s Q_{\beta(s)}(y) + \mathcal{H}_a Q_\beta = & \sum_{j=0}^{k} \left[(b_j)_s \eta_{b,j} + b_s \frac{b_j}{b} b \partial_b \eta_{b,j} + b b_j \lambda_{b,j} \eta_{b,j} + (a-b) b_j \Lambda \eta_{b,j}\right] \\ = & \sum_{j=0}^{k} \left\{ [(b_j)_s + b b_j \lambda_{b,j}] \eta_{b,j} + (a-b) b_j [\Lambda \eta_{b,j} - 2b \partial_b \eta_{b,j}] + \frac{b_j}{b} b \partial_b \eta_{b,j} \Phi \right\} \\ = & \sum_{j=0}^{k} \left\{ \left[(b_j)_s + b b_j \lambda_{b,j} + \frac{2(a-b)b_j}{|\mathrm{log} b|}\right] \eta_{b,j} + \frac{j b_j \Phi}{b} [\eta_{b,j} - \eta_{b,j-1}] \right. \\ + & \left. (a-b) b_j \left[\Lambda \eta_{b,j} - 2b \partial_b \eta_{b,j} - \frac{2}{|\mathrm{log} b|} \eta_{b,j}\right] + \frac{b_j}{b} \Phi \left[b \partial_b \eta_{b,j} - j(\eta_{b,j} - \eta_{b,j-1})\right] \right\}. \end{aligned}$$

The bounds (2.82), (2.84), (3.25), (3.30) now yield (3.34). □

Remark 3.7. The presence of the $|\log b|$ in the denominator on the right-hand side of (3.34) makes it a true error term with respect to our bootstrap regime, and this term is one of the leading order errors when closing the energy estimates in sections 3.3 and 3.4.

3.3. **Modulation equations.** From (3.3), (3.31) we obtain the equation satisfied by the perturbation ε :

$$\partial_s\varepsilon + \mathcal{H}_a\varepsilon = \mathcal{F} \tag{3.35}$$

where

$$\mathcal{F} = -\mathrm{Mod} - \Psi. \tag{3.36}$$

The nonlinear decomposition and the orthogonality conditions (3.14) generate a differential equation for the modulation vector $\beta = (b_j)_{0\leq j\leq k}$ in the setting of the bootstrap lemma 3.2 which we now compute exactly.

Lemma 3.8 (Modulation equations for b_j)**.** 1. $k = 0$*: the b law is given by:*

$$\left|b_s + \frac{2b^2}{|\log b|}\right| \lesssim \frac{b^2}{|\log b|^2} \tag{3.37}$$

2. $k \geq 1$*: the modulation dynamical system for the vector $(\tilde{b}_j)_{0\leq j\leq k}$ is given by*

$$\begin{aligned}&\left|(\tilde{b}_k)_s + \frac{1}{s}\left[\tilde{b}_k + (k+1)\sum_{j=0}^{k}\tilde{b}_j\right]\right| + \left|(\tilde{b}_{k-1})_s + \frac{1}{s}\left[\frac{k-1}{k}\tilde{b}_{k-1} - (k+1)\sum_{j=0}^{k}\tilde{b}_j\right]\right|\\ &+ \sum_{j=0}^{k-2}\left|(\tilde{b}_j)_s + \frac{j}{ks}\tilde{b}_j\right| \lesssim \frac{b^2}{|\log b|^2} + \frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\log b|}.\end{aligned} \tag{3.38}$$

Remark 3.9. The constants in Lemma 3.8 are independent of D, K, see also Remark 3.13. We need to keep track of the coupling between the modes in (3.38) in order to study the linearized system close to b_e^j and close the shooting argument, see (3.59).

Proof of Lemma 3.8. This lemma follows from the orthogonality conditions (3.14) and the boundary conditions of lemma 3.5.

step 1 Computation of Mod. The Mod estimate follows from the sharp choice of orthogonality conditions (3.14). Indeed, for any $0 \leq j \leq k$, we take the scalar product of (3.35) with $\eta_{b,j}$ and use the orthogonality condition (3.14) to compute:

$$-(\varepsilon, \partial_s\eta_{b,j})_b + \frac{b_s}{2}(\varepsilon, |y|^2\eta_{b,j})_b = (\mathcal{F}, \eta_{b,j}) - (a-b)(\Lambda\varepsilon, \eta_{b,j}).$$

We now intergrate by parts and use (3.14) again to compute:

$$\begin{aligned}&-(\varepsilon, \partial_s\eta_{b,j})_b + \frac{b_s}{2}(\varepsilon, |y|^2\eta_{b,j})_b + (a-b)(\Lambda\varepsilon, \eta_{b,j})\\ =\ & (\varepsilon, -\frac{b_s}{b}b\partial_b\eta_{b,j} + \frac{b_s}{2}|y|^2\eta_{b,j} + (a-b)[-\Lambda\eta_{b,j} + by^2\eta_{b,j}])\\ =\ & (\varepsilon, \frac{\Phi}{b}\left[-b\partial_b\eta_{b,j} + \frac{by^2}{2}\eta_{b,j}\right] + (a-b)[2b\partial_b\eta_{b,j} - \Lambda\eta_{b,j}])\end{aligned}$$

We evaluate all terms in the above expression. From (2.79), (2.80), (A.10):

$$|(\varepsilon, \frac{\Phi}{b}\left[-b\partial_b\eta_{b,j} + \frac{by^2}{2}\eta_{b,j}\right])| \lesssim \frac{|\Phi|}{b}\|\varepsilon\|_b\frac{|\log b|}{\sqrt{b}} \lesssim \frac{|\Phi||\log b|}{b^2}\frac{\sqrt{\mathcal{E}}}{\sqrt{b}}.$$

Similarily from (2.82), (2.83), and (3.30):

$$|(\varepsilon,(a-b)[2b\partial_b\eta_{b,j}-\Lambda\eta_{b,j}])|\lesssim\frac{|b-a|}{\sqrt{b}}\|\varepsilon\|_b\lesssim\frac{\sqrt{\mathcal{E}}}{\sqrt{b}}.$$

We now estimate the $\mathcal{F}$ terms given by (3.36). From (3.34), (2.83):

$$|(\Psi,\eta_{b,j})_b|\lesssim\|\Psi\|_b\|\eta_{b,j}\|_b\lesssim\left[\frac{b^{\frac{3}{2}}}{|\mathrm{log}b|}+\frac{|\Phi|}{\sqrt{b}}\right]\frac{|\mathrm{log}b|}{\sqrt{b}}\lesssim b+\frac{|\Phi||\mathrm{log}b|}{b}.$$

The collection of above bounds together with (2.83) and (3.36) yields

$$\frac{|(\mathrm{Mod},\eta_{\mathrm{b,j}})_{\mathrm{b}}|}{(\eta_{b,j},\eta_{b,j})_b}\lesssim\frac{b}{|\mathrm{log}b|^2}\left[b+\frac{|\Phi||\mathrm{log}b|}{b}+\left[1+\frac{|\Phi||\log b|}{b^2}\right]\frac{\sqrt{\mathcal{E}}}{\sqrt{b}}\right].\tag{3.39}$$

We now argue differently depending on k.

step 2 Case $k=0$. In this case, $b=b_0$ and thus from (3.26):

$$\Phi=(b_0)_s+2b_0(a-b_0)=b_s+O\left(\frac{b^2}{|\log b|^2}+\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\log b|}\right).$$

This together with (3.25) implies the bound:

$$|\Phi|\lesssim\frac{b^2}{|\log b|}+|b_s|\tag{3.40}$$

and thus (3.39), (3.32), and (3.26) imply:

$$|b_s+b^2\lambda_{b,0}|\lesssim\frac{b^2}{|\mathrm{log}b|^2}+\frac{|b_s|}{|\mathrm{log}b|}+\left[1+\frac{|b_s||\log b|}{b^2}\right]\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\mathrm{log}b|^2}.\tag{3.41}$$

Using the rough bound (3.25), this gives:

$$\left|b_s\left[1+O\left(\frac{1}{|\mathrm{log}b|}\right)\right]+\frac{2b^2}{|\mathrm{log}b|}\left[1+O\left(\frac{1}{|\mathrm{log}b|}\right)\right]\right|\lesssim\frac{b^2}{|\mathrm{log}b|^2}\tag{3.42}$$

and (3.37) follows.

step 3 Case $k\geq 1$. In this case, we have from (3.9):

$$|b_s|\lesssim b^2$$

and we therefore need to estimate Φ:

$$\begin{aligned}\Phi &= b_s+2b(a-b)=b_s+2b\left[a-\sum_{j=0}^k b_j\left(1+\frac{2\alpha_j}{|\mathrm{log}b|}\right)\right]\\ &+ 2b\sum_{j=0}^k(b_j-b_j^e)\left(1+\frac{2\alpha_j}{|\mathrm{log}b|}\right)+2b\left[b_k^e\left(1+\frac{2\alpha_j}{|\mathrm{log}b|}\right)-b\right]\\ &= 2b\sum_{j=0}^k\tilde{b}_j+O\left(\frac{b^2}{|\mathrm{log}b|^2}+\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\log b|}\right)\end{aligned}\tag{3.43}$$

where we used (3.26), (3.12) and the bootstrap bounds (3.23), (4.13) in the last step. This implies in particular the rough bound

$$|\Phi|\leq\frac{b^2}{|\log b|}\tag{3.44}$$

From (3.39) it follows that:

$$\frac{|(\mathrm{Mod},\eta_{\mathrm{b,j}})_{\mathrm{b}}|}{(\eta_{b,j},\eta_{b,j})_b}\lesssim\frac{b^2}{|\log b|^2}+\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\log b|^2}\lesssim\frac{b^2}{|\log b|^2}. \tag{3.45}$$

We now recall (3.11) and compute from (3.26), (3.12):

$$\begin{aligned} a &= \sum_{j=0}^k(b_j^e+\tilde{b}_j)\left[1+\frac{2\alpha_j}{|\log b|}\right]+O\left(\frac{b}{|\log b|^2}+\frac{\sqrt{\mathcal{E}}}{|\log b|\sqrt{b}}\right)\\ &= a^e+\sum_{j=0}^k\tilde{b}_j+O\left(\frac{b}{|\log b|^2}+\frac{\sqrt{\mathcal{E}}}{|\log b|\sqrt{b}}\right). \end{aligned}\tag{3.46}$$

We now use (3.43), (3.12), (3.46) to compute explicitly:

$$\begin{aligned} &(b_k)_s+bb_k\lambda_{b,k}+\frac{2(a-b)b_k}{|\log b|}+\frac{kb_k}{b}\Phi\\ =\;&(b_k^e+\tilde{b}_k)_s+b(b_k^e+\tilde{b}_k)\left[2k+\frac{2}{|\log b|}+O\left(\frac{1}{|\log b|^2}\right)\right]+\frac{2(a^e-b)(b_k^e+\tilde{b}_k)}{|\log b|}\\ &+\frac{2(a-a^e)b_k}{|\log b|}+k(b_k^e+\tilde{b}_k)\left[2\sum_{j=0}^k\tilde{b}_j+O\left(\frac{b}{|\log b|^2}+\frac{\sqrt{\mathcal{E}}}{\sqrt{b}|\log b|}\right)\right]\\ =\;&(\tilde{b}_k)_s+\frac{1}{s}\left[\tilde{b}_k+(k+1)\sum_{j=0}^k\tilde{b}_j\right]+O\left(\frac{b^2}{|\log b|^2}+\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\log b|}\right). \end{aligned}$$

Similarily for $j=k-1$:

$$\begin{aligned} &(b_{k-1})_s+bb_{k-1}\lambda_{b,k-1}+\frac{2(a-b)b_{k-1}}{|\log b|}+\frac{(k-1)b_{k-1}-kb_k}{b}\Phi\\ =\;&(\tilde{b}_{k-1})_s+b\tilde{b}_{k-1}\left[2(k-1)+\frac{2}{|\log b|}+O\left(\frac{1}{|\log b|^2}\right)\right]+\frac{2(a^e-b)\tilde{b}_{k-1}}{|\log b|}\\ +\;&\frac{2(a-a^e)\tilde{b}_{k-1}}{|\log b|}+\left[(k-1)\tilde{b}_{k-1}-k(b_k^e+\tilde{b}_k)\right]\left[2\sum_{j=0}^k\tilde{b}_j+O\left(\frac{b}{|\log b|^2}+\frac{\sqrt{\mathcal{E}}}{\sqrt{b}|\log b|}\right)\right]\\ =\;&(\tilde{b}_{k-1})_s+\frac{1}{s}\left[\frac{k-1}{k}\tilde{b}_{k-1}-(k+1)\sum_{j=0}^k\tilde{b}_j\right]+O\left(\frac{b^2}{|\log b|^2}+\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\log b|}\right). \end{aligned}$$

Finally for $0\leq j\leq k-2$:

$$\begin{aligned} &(b_j)_s+bb_j\lambda_{b,j}+\frac{2(a-b)b_j}{|\log b|}+\frac{jb_j-(j+1)b_{j+1}}{b}\Phi\\ =\;&(\tilde{b}_j)_s+\frac{j}{ks}\tilde{b}_j+O\left(\frac{b^2}{|\log b|^2}+\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\log b|}\right). \end{aligned}$$

Injecting the above bounds into (3.45) yields (3.38). □

3.4. **Energy bound.** We now arrive at the second main feature of the analysis which is the derivation of suitable energy bounds for ε. The key here is the dissipation embedded in the problem and its geometry which feeds back into the energy estimates through the boundary conditions. A careful analysis of this interaction will allow us to close the energy estimates.

Proposition 3.10 (Energy bound)**.** *There holds the pointwise control:*
1. for $k=0$*:*
$$\frac{1}{2}\frac{d}{ds}\left\{\mathcal{E}+O\left(\frac{b^3}{|\log b|^2}\right)\right\}+cb\mathcal{E}\lesssim\frac{b^4}{|\log b|^2};\tag{3.47}$$
2. for $k\geq 1$*:*
$$\frac{1}{2}\frac{d}{ds}\left\{\mathcal{E}+O\left(\frac{b^3}{|\log b|^{5/4}}\right)\right\}+[3k+c]\,b\mathcal{E}\lesssim\frac{Kb^4}{|\log b|}\tag{3.48}$$
for some universal constant $c>0$*.*

Remark 3.11. The sharp coercivity constant $3k$ in (3.48) which follows from the sharp Poincaré estimate (A.3) is essential to close the energy bound, see (3.58).

Proof of Proposition 3.10. We compute the energy identity for $\mathcal{E}$ and estimate all terms.

step 1 Algebraic energy identity. Recalling from (3.6) that $\varepsilon_2=\mathcal{H}_b\varepsilon$, it follows from (3.35):
$$\partial_s\varepsilon_2+\mathcal{H}_a\varepsilon_2=[\partial_s,\mathcal{H}_b]\varepsilon+[\mathcal{H}_a,\mathcal{H}_b]\varepsilon+\mathcal{H}_b\mathcal{F}.$$
To compute the commutators $[\partial_s,\mathcal{H}_b]$, $[\mathcal{H}_a,\mathcal{H}_b]$ we use
$$[\Delta,\Lambda]=2\Delta\tag{3.49}$$
which yields:
$$\begin{aligned}&[\partial_s,\mathcal{H}_b]\varepsilon+[\mathcal{H}_a,\mathcal{H}_b]\varepsilon=b_s\Lambda\varepsilon+[\mathcal{H}_b+(a-b)\Lambda,\mathcal{H}_b]\varepsilon=b_s\Lambda\varepsilon+(a-b)[\Lambda,-\Delta]\\ =\ &b_s\Lambda\varepsilon+2(a-b)\Delta\varepsilon=(b_s+2b(a-b))\Lambda\varepsilon-2(a-b)[-\Delta\varepsilon+b\Lambda\varepsilon]\\ =\ &\Phi\Lambda\varepsilon+2(b-a)\varepsilon_2.\end{aligned}$$
Hence the ε_2 equation:
$$\partial_s\varepsilon_2+\mathcal{H}_a\varepsilon_2=\Phi\Lambda\varepsilon+2(b-a)\varepsilon_2+\mathcal{H}_b\mathcal{F}.\tag{3.50}$$
We now compute the modified energy identity:
$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\mathcal{E}&=\frac{1}{2}\frac{d}{ds}\int_{y\geq1}\varepsilon_2^2e^{-\frac{by^2}{2}}ydy=-\frac{b_s}{4}\int_{y\geq1}y^2|\varepsilon_2|^2e^{-\frac{b|y|^2}{2}}ydy+(\partial_s\varepsilon_2,\varepsilon_2)_b\\ &=-\frac{b_s}{4}\|y\varepsilon_2\|_b^2+(\Phi\Lambda\varepsilon+2(b-a)\varepsilon_2+\mathcal{H}_b\mathcal{F}-\mathcal{H}_a\varepsilon_2,\varepsilon_2)_b.\end{aligned}$$
We carefully integrate by parts to compute:
$$\begin{aligned}&-\int_{y\geq1}\varepsilon_2\mathcal{H}_a\varepsilon_2e^{-\frac{by^2}{2}}ydy=-\int_{y\geq1}\varepsilon_2[\mathcal{H}_b\varepsilon_2+(a-b)\Lambda\varepsilon_2]e^{-\frac{by^2}{2}}ydy\\ =\ &\int_{y\geq1}\partial_y(\rho_by\partial_y\varepsilon_2)\varepsilon_2dy+(b-a)\int\varepsilon_2y\partial_y\varepsilon_2e^{-\frac{by^2}{2}}ydy\\ =\ &-\rho_b(1)\varepsilon_2(1)\partial_y\varepsilon_2(1)-\int_{y\geq1}|\partial_y\varepsilon_2|^2e^{-\frac{by^2}{2}}ydy\\ &-\frac{b-a}{2}\rho_b(1)\varepsilon_2^2(1)-\frac{b-a}{2}\int_{y\geq1}\varepsilon_2^2\left[2-by^2\right]e^{-\frac{by^2}{2}}ydy\\ =\ &-\|\partial_y\varepsilon_2\|_b^2+(a-b)\|\varepsilon_2\|_2^2-\frac{b(a-b)}{2}\|y\varepsilon_2\|_b^2-\rho_b(1)\varepsilon_2(1)\left[\partial_y\varepsilon_2+\frac{b-a}{2}\varepsilon_2\right](1).\end{aligned}$$

This yields the algebraic energy identity:

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\mathcal{E} &= -\|\partial_y\varepsilon_2\|_b^2-(a-b)\|\varepsilon_2\|_2^2-\frac{\Phi}{4}\|y\varepsilon_2\|_b^2-\rho_b\varepsilon_2\left[\partial_y\varepsilon_2+\frac{b-a}{2}\varepsilon_2\right](1)\\ &+ \Phi(\Lambda\varepsilon,\varepsilon_2)_b+(\mathcal{H}_b\mathcal{F},\varepsilon_2)_b.\end{aligned}\tag{3.51}$$

We now estimate all terms in the right-hand side of (3.51).

step 2 Nonlinear estimates. From (A.10), (3.44), (3.25):

$$|\Phi||(\Lambda\varepsilon,\varepsilon_2)_b|\lesssim\frac{b^2}{|\mathrm{log}b|}\frac{\sqrt{\mathcal{E}}}{b}\sqrt{\mathcal{E}}\lesssim b\frac{\mathcal{E}}{|\mathrm{log}b|}.$$

Moreover from (3.14) $(\mathcal{H}_b\mathrm{Mod},\varepsilon_2)_{\mathrm{b}}=0$, and from (3.34), (3.44):

$$|(\mathcal{H}_b\Psi,\varepsilon_2)_b\lesssim b\sqrt{\mathcal{E}}\frac{b^{\frac32}}{|\mathrm{log}b|}.$$

We now estimate from (A.12), (3.44), (3.27), (3.30):

$$|\Phi|\|y\varepsilon_2\|_b^2\lesssim\frac{b^2}{|\mathrm{log}b|}\left[\frac{\|\partial_y\varepsilon_2\|_b^2}{b^2}+b^3\right]\lesssim\frac{\|\partial_y\varepsilon_2\|_b^2}{|\mathrm{log}b|}+\frac{b^4}{|\mathrm{log}b|^2}.$$

It now remains to treat the boundary term in (3.51) and we argue differently depending on k.

step 3 Conclusion for $k=0$. We compute from (3.28), (3.37):

$$\begin{aligned}\partial_y\varepsilon_2(1) &= -a_s-\frac{2b^2}{|\mathrm{log}b|}+O\left(\frac{b^2}{|\mathrm{log}b|^2}\right)=-(a-b)_s-(b_s+\frac{2b^2}{|\mathrm{log}b|})+O\left(\frac{b^2}{|\mathrm{log}b|^2}\right)\\ &= -(a-b)_s+O\left(\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\mathrm{log}b|}+\frac{b^2}{|\mathrm{log}b|^2}\right)\end{aligned}$$

and hence using (3.27):

$$\begin{aligned}&\rho_b(1)\varepsilon_2(1)\left[\partial_y\varepsilon_2+\frac{b-a}{2}\varepsilon_2\right](1)\\ =\ & e^{-\frac{b}{2}}a(a-b)\left[-(a-b)_s+\frac{a(a-b)^2}{2}+O\left(\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\mathrm{log}b|}+\frac{b^2}{|\mathrm{log}b|^2}\right)\right]\\ =\ & e^{-\frac{b}{2}}\left[-(a-b)^2(a-b)_s-b(a-b)(a-b)_s\right]+O\left(b\frac{b^{\frac32}\sqrt{\mathcal{E}}}{|\mathrm{log}b|}+\frac{b^4}{|\mathrm{log}b|^2}\right)\\ =\ & -\frac{d}{ds}\left\{e^{-\frac{b}{2}}\frac{(a-b)^3}{6}+e^{-\frac{b}{2}}\frac{b(a-b)^2}{2}\right\}-\frac{b_se^{-\frac{b}{2}}}{2}\frac{(a-b)^3}{6}\\ & +e^{-\frac{b}{2}}\frac{b(a-b)^2}{2}\left[-\frac{b_s}{2}+\frac{b_s}{b}\right]+O\left(b\frac{b^{\frac32}\sqrt{\mathcal{E}}}{|\mathrm{log}b|}+\frac{b^4}{|\mathrm{log}b|^2}\right)\\ =\ & -\frac{d}{ds}\left\{e^{-\frac{b}{2}}\frac{(a-b)^3}{6}+e^{-\frac{b}{2}}\frac{b(a-b)^2}{2}\right\}+O\left(b\frac{b^{\frac32}\sqrt{\mathcal{E}}}{|\mathrm{log}b|}+\frac{b^4}{|\mathrm{log}b|^2}+\frac{b^2(a-b)^2}{|\mathrm{log}b|}\right).\end{aligned}$$

We now observe from (3.26), (3.25) that

$$|a-b|\lesssim\frac{b}{|\mathrm{log}b|},\tag{3.52}$$

and hence the collection of above bounds yields the control:

$$\frac{1}{2}\frac{d}{ds}\left\{\mathcal{E}+O\left(\frac{b^3}{|\log b|^2}\right)\right\}=-\|\partial_y\varepsilon_2\|_b^2+O\left(\frac{\|\partial_y\varepsilon_2\|_b^2+\|\varepsilon_2\|_b^2}{|\log b|}+b\frac{b^{\frac{3}{2}}\sqrt{\mathcal{E}}}{|\log b|}+\frac{b^4}{|\log b|^2}\right).$$

We now inject (3.27), (A.11) with $k=0$ and (3.47) follows.

step 4 Conclusion for $k\geq 1$. Let

$$\tilde{a}=a^e+\sum_{j=0}^{k}\tilde{b}_j$$

be the leading order part of a, see (3.46). Then from (3.38), (4.13), (3.12):

$$\begin{aligned}
&\tilde{a}_s+\sum_{j=0}^{k}\lambda_{b,j}bb_j\left[1+\frac{2\alpha_j}{|\log b|}\right]\\
&=(a^e)_s+\sum_{j=0}^{k}(\tilde{b}_j)_s+\sum_{j=0}^{k}\lambda_{b,j}b(b_j^e+\tilde{b}_j)\left[1+\frac{2\alpha_j}{|\log b|}\right]\\
=&\left(a^e-b_k^e\left[1+\frac{2\alpha_k}{|\log b|}\right]\right)_s+\left(1+\frac{2\alpha_k}{|\log b|}\right)[(b_k^e)_s+\lambda_{b,j}bb_k^e]+O\left(\frac{b^2}{|\log b|^2}\right)\\
+&O\left(\frac{Kb^2}{|\log b|^{3/2}}+\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\log b|}\right)=-2\frac{(a^e-b)b_e^k}{|\log b|}+O\left(\frac{Kb^2}{|\log b|^{3/2}}+\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\log b|}\right)\\
=&-\frac{k+1}{2ks^2|\log s|}+O\left(\frac{Kb^2}{|\log b|^{3/2}}+\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\log b|}\right).
\end{aligned}$$

We therefore estimate using Lemma 3.5:

$$\begin{aligned}
&-\rho_b\varepsilon_2\left[\partial_y\varepsilon_2+\frac{b-a}{2}\varepsilon_2\right](1)\\
=&-e^{-\frac{b}{2}}a(a-b)\left[a_s+\sum_{j=0}^{k}\lambda_{b,j}bb_j\left[1+\frac{2\alpha_j}{|\log b|}\right]+O\left(\frac{b^2}{|\log b|^2}\right)\right]\\
=&-e^{-\frac{b}{2}}a(a-b)\left[(a-\tilde{a})_s-\frac{k+1}{2ks^2\log s}+O\left(\frac{Kb^2}{|\log b|^{3/2}}+\frac{\sqrt{b}\sqrt{\mathcal{E}}}{|\log b|}\right)\right]\\
=&-e^{-\frac{b}{2}}a(a-b)(a-\tilde{a})_s+e^{-\frac{b}{2}}a(a-b)\frac{k+1}{2ks^2\log s}+O\left(\frac{Kb^4}{|\log b|^{3/2}}+\frac{bb^{\frac{3}{2}}\sqrt{\mathcal{E}}}{|\log b|}\right)\\
=&-e^{-\frac{b}{2}}(a-\tilde{a})_s\left[(a-\tilde{a})^2+(a-\tilde{a})(2\tilde{a}-b)+\tilde{a}(\tilde{a}-b)\right]\\
&+O(\frac{b^4}{|\log b|})+O\left(\frac{Kb^4}{|\log b|^{3/2}}+\frac{bb^{\frac{3}{2}}\sqrt{\mathcal{E}}}{|\log b|}\right)\\
=&-\frac{d}{ds}\left\{-e^{-\frac{b}{2}}\left[\frac{(a-\tilde{a})^3}{3}+\frac{(a-\tilde{a})^2}{2}(2\tilde{a}-b)+(a-\tilde{a})\tilde{a}(\tilde{a}-b)\right]\right\}\\
&+O\left(\frac{b^4}{|\log b|}+\frac{Kb^4}{|\log b|^{3/2}}+\frac{bb^{\frac{3}{2}}\sqrt{\mathcal{E}}}{|\log b|}+b^3|a-\tilde{a}|\right).
\end{aligned}$$

We have the bound from (3.46):

$$a - \tilde{a} \quad = \quad O\left(\frac{b}{|\log b|} + \frac{\sqrt{\mathcal{E}}}{|\log b|\sqrt{b}}\right).$$

Injecting the above bounds into (3.51) and using the rough bound (3.25) we obtain the bound:

$$\frac{1}{2}\frac{d}{ds}\left\{\mathcal{E} + O\left(\frac{b^3}{|\log b|}\right)\right\} = -\|\partial_y \varepsilon_2\|_b^2 - (a-b)\|\varepsilon_2\|_2^2$$
$$+ O\left(\frac{\|\partial_y \varepsilon_2\|_b^2 + \|\varepsilon_2\|_b^2}{|\log b|} + b\frac{b^{\frac{3}{2}}\sqrt{\mathcal{E}}}{|\log b|} + \frac{b^4}{|\log b|}\right).$$

We now estimate from (3.46):

$$a - b = \frac{1}{2s} + O\left(\frac{b}{|\log b|}\right) = kb + O\left(\frac{b}{|\log b|}\right) \tag{3.53}$$

and (3.48) now follows from (A.11) with (3.27). □

3.5. **Proof of Proposition 3.2.** We are now in position to give a sharp description of the singularity formation for our set of initial data. The key is to close the bootstrap bounds of Proposition 3.2. We distinguish the cases $k = 0$ and $k \geq 1$.

step 1 Closing the bootstrap bounds. Our goal is to show that the bounds (3.21), (3.22) improve in the case $k = 0$ and similarly for the bounds (3.21), (3.23), (4.13) in the case $k \geq 1$. The improvement of the energy bound (3.21) will follow from proposition 3.10, while the bounds (3.23), (4.13) will be improved for a suitable set of initial data constructed via a topological argument.

$\underline{k = 0}$. First observe that (3.37) ensures

$$b_s < 0 \ \text{ and hence } \ b(s) < b_0 \leq b^*.$$

From (3.52),

$$\frac{\lambda_s}{\lambda} + b = b - a = O\left(\frac{b}{|\log b|}\right) \tag{3.54}$$

and hence:

$$\log\lambda(s) = -\int_0^s b\left[1 + O\left(\frac{1}{|\log b|}\right)\right] d\sigma < +\infty \ \text{ implies } \ \lambda(s) > 0. \tag{3.55}$$

We now rewrite (3.48) (with $k = 0$) as

$$\frac{d}{ds}\left\{\frac{1}{2}\mathcal{E} + O\left(\frac{b^3}{|\log b|^2}\right)\right\} + bc\left[\mathcal{E} + O\left(\frac{b^3}{|\log b|^2}\right)\right] \lesssim \frac{b^4}{|\log b|^2}$$

with $c > 0$. Using (3.54) we obtain:

$$\frac{d}{ds}\left\{\frac{1}{\lambda^c}\left[\frac{1}{2}\mathcal{E} + O\left(\frac{b^3}{|\log b|^2}\right)\right]\right\} \lesssim \frac{b^4}{\lambda^c|\log b|^2}. \tag{3.56}$$

We now integrate in time. To evaluate the right hand side, we integrate by parts using (3.37):

$$\int_0^s \frac{b^4}{\lambda^c|\log b|^2} d\sigma = \int_0^s \left[-\frac{\lambda_s}{\lambda}\frac{b^3}{\lambda^c|\log b|^2} + O\left(\frac{b^4}{|\log b|^3}\right)\right] d\sigma$$
$$= \left[\frac{1}{c}\frac{b^3}{\lambda^c|\log b|^2}\right]_0^s - \frac{1}{c}\int_0^s \frac{b_s}{\lambda^c}\left[\frac{3b^2}{|\log b|^2} + \frac{2b^2}{|\log b|^3}\right] d\sigma + \int_0^s O\left(\frac{b^4}{\lambda^c|\log b|^3}\right) d\sigma.$$

Using the smallness of b we get:

$$\int_0^s \frac{b^4}{\lambda^c|\log b|^2}d\sigma \lesssim \frac{b^3}{\lambda^c|\log b|^2}(s).$$

Hence (3.1), (3.15) and the time integration of (3.56) ensure:

$$\mathcal{E}(s) \lesssim \frac{b^3}{|\log b|^2}(s) + \lambda^c(s)\left[\mathcal{E}(0) + \frac{b_0^3}{|\log b_0|^2}\right] \lesssim \frac{b^3}{|\log b|^2}(s) + \lambda^c(s)\frac{b_0^3}{|\log b_0|^2}. \tag{3.57}$$

We moreover estimate from (3.37):

$$\frac{d}{ds}\left\{\frac{b^3}{\lambda^c|\log b|^2}\right\} = \frac{b^3}{\lambda^c|\log b|^2}\left[\frac{3b_s}{b} + \frac{2b_s}{b|\log b|} - c\frac{\lambda_s}{\lambda}\right] = \frac{b^3}{\lambda^c|\log b|^2}\left[cb + O\left(\frac{b}{|\log b|^2}\right)\right] > 0$$

and hence using (3.1) again:

$$\lambda^c(s)\frac{b_0^3}{|\log b_0|^2} \leq \lambda^c(s_0)\frac{b^3}{|\log b|^2}(s)$$

which together with (3.55) implies $b(s) > 0$ and closes the bound (3.22). Injecting this into (3.57) improves the energy bound (3.21) for D universal large enough, which concludes the proof of Proposition 3.2 for $k = 0$.

$k \geq 1$. This case requires a shooting argument to build the nonlinear manifold of perturbations $(V_j)_{0\leq j\leq k-1}$. We first rewrite (3.48) using (3.9):

$$\frac{d}{ds}\left\{\mathcal{E} + O\left(\frac{b^3}{|\log b|}\right)\right\} + \left[3 + \frac{c}{k}\right]\frac{1}{s}\mathcal{E} \lesssim \frac{K}{s^4|\log s|}.$$

Using (3.15), an integration-in-time yields:

$$\begin{aligned}\mathcal{E}(s) &\leq \frac{s_0^{3+\frac{c}{k}}}{s^{3+\frac{c}{k}}}\left[\mathcal{E}_0 + \frac{b_0^3}{|\log b_0|}\right] + \frac{b^3}{|\log b|} + \frac{1}{s^{3+\frac{c}{k}}}\int_{s_0}^s \frac{K\sigma^{3+\frac{c}{k}}}{\sigma^4|\log \sigma|}d\sigma \\ &\lesssim \frac{K}{s^3(\log s)} \lesssim K\frac{b^3}{|\log b|}.\end{aligned} \tag{3.58}$$

This means that there exists a $\tilde{C} > 0$ universal large enough such that if $D = \tilde{C}K$, the bound (3.21) gets improved, and we assume it now. We inject this relation into (3.38) and conclude:

$$\begin{aligned}&\left|(\tilde{b}_k)_s + \frac{1}{s}\left[\tilde{b}_k + (k+1)\sum_{j=0}^k \tilde{b}_j\right]\right| + \left|(\tilde{b}_{k-1})_s + \frac{1}{s}\left[\frac{k-1}{k}\tilde{b}_{k-1} - (k+1)\sum_{j=0}^k \tilde{b}_j\right]\right| \\ &+ \sum_{j=0}^{k-2}\left|(\tilde{b}_j)_s + \frac{j}{ks}\tilde{b}_j\right| \lesssim \frac{\sqrt{K}b^2}{|\log b|^{3/2}} \lesssim \frac{\sqrt{K}}{s^2(\log s)^{3/2}}.\end{aligned}$$

Equivalently using the change of variables (3.16):

$$\begin{aligned}&\left|(V_k)_s + \frac{(k+1)\sum_{j=0}^k V_j}{s}\right| + \left|(V_{k-1})_s + \frac{1}{s}\left[-\frac{1}{k}V_{k-1} - (k+1)\sum_{j=0}^k V_j\right]\right| \\ &+ \sum_{j=0}^{k-2}\left|(V_j)_s + \frac{j-k}{ks}V_j\right| \lesssim \frac{\sqrt{K}}{s}.\end{aligned} \tag{3.59}$$

The bootstrap bound (4.13) implies that $|V_j| \le \delta K$, $j = 0, \dots k-2$. Therefore, from the first two bounds in (3.59) we conclude that for a sufficiently large K the following bound holds

$$\left|(V_k)_s + \frac{(k+1)}{s}(V_k + V_{k-1})\right| + \left|(V_{k-1})_s - \frac{k+1}{s}(V_k + (1+d_k)V_{k-1})\right| \lesssim \frac{\delta K}{s},$$

where we remind the reader that $d_k = \frac{1}{k(k+1)}$. Recalling the definition (3.17) of the matrix A_k, the above inequalities can be succinctly rewritten in the form

$$\partial_s \begin{pmatrix} V_k \\ V_{k-1} \end{pmatrix} = \frac{k+1}{s} A_k \begin{pmatrix} V_k \\ V_{k-1} \end{pmatrix} + O\left(\frac{\delta K}{s}\right),$$

which in turn leads to

$$\left|(W_k)_s + \frac{(k+1)\mu_1^k}{s} W_k\right| + \left|(W_{k-1})_s + \frac{(k+1)\mu_2^k}{s} W_{k-1}\right| \lesssim \frac{\delta K}{s}, \tag{3.60}$$

where W_k, W_{k-1} are defined in (3.18) and $\mu_2^k < 0 < \mu_1^k$ are the eingenvalues of A_k. This first yields the control of the stable direction W_k, since after integrating-in-time the first bound in (3.60), we arrive at

$$|W_k(s)| \le |W_k(0)| \frac{s_0^{(k+1)\mu_1^k}}{s^{(k+1)\mu_2^k}} + \frac{1}{s^{(k+1)\mu_1^k}} \int_{s_0}^{s} \frac{\delta K \sigma^{(k+1)\mu_1^k}}{\sigma} d\sigma \le 1 + C\delta K,$$

where we used (3.19) and the positivity of μ_1^k. This improves (3.23) for K sufficiently large and $\delta < \frac{1}{2C}$. We now argue by contradiction and assume that for all $(\frac{V_0}{\delta}, \dots, \frac{V_{k-2}}{\delta}, W_{k-1}) \in B_K(\mathbb{R}^{d-1})$, the boostrap time s^* is finite, so that from (4.13):

$$|W_{k-1}(s^*)|^2 + \sum_{j=0}^{k-2} \left|\frac{V_j(s^*)}{\delta}\right|^2 = K^2. \tag{3.61}$$

We claim that this contradicts the Brouwer fixed point theorem. Indeed, using (3.59), (3.60), and the strict negativity of μ_2^k:

$$\begin{aligned} & \frac{1}{2}\frac{d}{ds}\left\{ |W_{k-1}(s)|^2 + \sum_{j=0}^{k-2} \left|\frac{V_j(s)}{\delta}\right|^2 \right\}(s^*) \\ = \; & \frac{1}{s^*}\left[|\mu_2^k|(k+1)W_{k-1}^2(s^*) + \sum_{j=0}^{k-2} \frac{k-j}{\delta^2 k} V_j^2(s^*) + O\left(\delta K^2 + K^{3/2}\right)\right] \\ \ge \; & \frac{c}{s^*}\left[K^2 - C\delta K^2\right], \end{aligned}$$

for some universal constants $c, C > 0$. Hence

$$\frac{d}{ds}\left\{ |W_{k-1}(s^*)|^2 + \sum_{j=0}^{k-2} \left|\frac{V_j(s^*)}{\delta}\right|^2 \right\}(s^*) > 0 \tag{3.62}$$

for $0 < \delta \ll 1$ universal small enough in (4.13). Let

$$\tilde{V} = (\frac{V_0}{\delta}, \dots, \frac{V_{k-2}}{\delta}, W_{k-1}),$$

then this implies from standard from standard argument that the map

$$B_K(\mathbb{R}^{d-1}) \ni \tilde{V}(0) \mapsto s^*\left(\tilde{V}(0)\right)$$

is continuous, and hence the map

$$\begin{aligned} & B_K(\mathbb{R}^{d-1}) \to B_K(\mathbb{R}^{d-1}) \\ & \tilde{V}(0) \mapsto \tilde{V}\left[\tilde{s}^*(\tilde{V}(0))\right] \end{aligned}$$

is continuous and the identity on the boundary sphere $\mathbb{S}_{d-1}(K)$, a contradiction to Brouwer's fixed point theorem. This concludes the proof of Proposition 3.2 for $k \geq 1$.

Remark 3.12. Note that $(V_0(\varepsilon_0), \dots V_{k-2}(\varepsilon_0), W_{k-1}(\varepsilon_0))$ are by construction lying on a nonlinear codimension k manifold of initial data. The fact that the set of such initial data forms a Lipschitz manifold in the H^2 topology reduces to a local uniqueness problem in the class of solutions satisfying the a priori bounds of Proposition 3.2. Such a uniqueness problem has been recently solved in a related framework in the more complicated case of the wave equation [4] and the KdV equation [29], see also [**?**], and a completely analogous approach can be applied here. We therefore omit the details.

Remark 3.13. Note that the presence of $\sqrt{K}$ on the right-hand side of (3.59) is essential to the closure of the estimates. It originates from the bound (3.38), where we carefully tracked the constants and proved that only $\sqrt{\mathcal{E}}$ appears on the right-hand side of (3.38).

step 2 Global H^2 control. From Proposition 3.2, the solution remains in the bootstrap regime of Proposition 3.2 as long as it exists in H^2 which requires: $\forall s \geq 0$,

$$\|u(s)\|_{L^2(|x|\geq\lambda(s))} + \|\nabla u(s)\|_{L^2(|x|\geq\lambda(s))} + \|\Delta u(s)\|_{L^2(|x|\geq\lambda(s))} < +\infty \tag{3.63}$$

and

$$\lambda(s) > 0. \tag{3.64}$$

In the case $k = 0$, the positivity of λ follows from the time integration of (3.37) which implies

$$|\mathrm{log}\lambda(s)| \lesssim \int_0^s b\left(1 + O(\frac{1}{|\mathrm{log}b|})\right) d\sigma < +\infty,$$

while in the case $k \geq 1$ we use $a = -\lambda_s/\lambda$ and the estimate (3.53), which implies the above bound again. The global L^2-bound follows from the basic dissipation law satisfied by the solutions of (1.1):

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega(t))} + \|\nabla u\|^2_{L^2(\Omega(t))} = 0,$$

which immediately implies that $\|u(s)\|_{L^2(|x|\geq\lambda(s))} < \infty$. The global $\dot{H}^1$-bound follows directly from the dissipation of the Dirichlet energy (1.3). For the global $\dot{H}^2$-bound, we take a cut off function $\chi = 0$ for $r \leq 1$ and $r = 1$ for $r \geq 2$, then the weighted control (3.21) ensures

$$\|\Delta u(s)\|_{L^2(\lambda(s)\leq r\leq 2)} < C(s) < +\infty \quad \text{for} \quad s \geq 0$$

since the exponential weight is uniformly bounded from below and above in $\lambda(s) \leq r \leq 2$. To obtain the bound in the region $r \geq 2$ we compute:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int \chi|\Delta u|^2 &= \int \chi\Delta\partial_t u\Delta u = \int \chi\Delta^2 u\Delta u \\ &= -\int \chi|\nabla\Delta u|^2 + \frac{1}{2}\int \Delta\chi|\Delta u|^2 \end{aligned} \tag{3.65}$$

and hence

$$\int \chi|\Delta u|^2(s) \lesssim \|\chi\Delta u(0)\|_{L^2}^2 + \int_0^s \frac{1}{\lambda^2(\sigma)}\|\Delta u(s)\|^2_{L^2(\lambda(\sigma)\leq r\leq 2)}d\sigma < +\infty.$$

Hence $s^* = +\infty$ which concludes the proof of Proposition 3.2.

3.6. **Proof of Theorem 1.1.** We are now in position to conclude the proof of Theorem 1.1.

step 1 Finite time melting. We claim that the solution melts in finite time with the law (1.4), (1.5) as a consequence of the time integration of the modulation equations.

case $k = 0$: From (3.37), (3.21), we obtain the following pointwise differential inequality for b :

$$b_s + \frac{2b^2}{|\log b|} = O\left(\frac{b^2}{|\log b|^2}\right). \tag{3.66}$$

We now follow [41, 44] to derive the melting speed of λ and sketch the proof for the sake of clarity. Multiplying (3.66) by $\frac{\log b}{b^2}$ we obtain

$$\frac{b_s \log b}{b^2} = 2 + O\left(\frac{1}{|\log b|}\right).$$

The primitive of $\frac{\log u}{u^2}$ is $-\frac{\log u}{u} - \frac{1}{u}$ and therefore

$$\frac{\log b}{b} + \frac{1}{b} = -2s + O(\int_0^s \frac{1}{|\log b|}\,d\tau)$$

which implies

$$\frac{\log s}{s} \lesssim b \lesssim \frac{\log s}{s}.$$

Hence:

$$b = -\frac{1+\log b}{2s}\left[1 + O\left(\frac{1}{s}\int_0^s \frac{1}{|\log b|}\,d\tau\right)\right] = -\frac{1+\log b}{2s}\left[1 + O\left(\frac{1}{|\log b|}\right)\right] \tag{3.67}$$

Taking the log yields

$$\log b = -\log 2 - \log s + \log(-\log b) + O\left(\frac{1}{|\log b|}\right)$$

which reinjected into (3.67) ensures:

$$b = \frac{\log s}{2s}\left[1 + O\left(\frac{\log\log s}{\log s}\right)\right], \quad \log b = \log\log s - \log 2 - \log s + O\left(\frac{\log\log s}{\log s}\right). \tag{3.68}$$

Injecting this into (3.67) again yields:

$$\begin{aligned} b &= -\frac{-\log s + \log\log s - \log 2 + 1 + O\left(\frac{\log\log s}{\log s}\right)}{2s}\left[1 + O\left(\frac{1}{\log s}\right)\right] \\ &= \frac{\log s}{2s} - \frac{\log\log s}{2s} + O\left(\frac{1}{s}\right). \end{aligned} \tag{3.69}$$

Recalling from (3.26) that

$$b = -\frac{\lambda_s}{\lambda} + O\left(\frac{b}{|\log b|^2}\right), \tag{3.70}$$

we conclude that

$$-(\log\lambda)_s = \frac{\log s}{2s} - \frac{\log\log s}{2s} + O(\frac{1}{s}),$$

which gives

$$-\log\lambda = \frac{1}{4}(\log s)^2 - \frac{1}{2}\log s\log\log s + O(\log s),$$

which in turn gives

$$-2\log(\lambda^2) = (\log s)^2\left[1 - 2\frac{\log\log s}{\log s} + O\left(\frac{1}{\log s}\right)\right].$$

This leads to

$$\begin{aligned}\sqrt{-2\log(\lambda^2)} &= \log s\left[1 - \frac{\log\log s}{\log s} + O\left(\frac{1}{\log s}\right)\right]\\ &= \log s - \log\log s + O(1)\end{aligned} \tag{3.71}$$

from which

$$e^{\sqrt{-2\log(\lambda^2)}} = \frac{s}{\log s}e^{O(1)}$$

and hence from (3.70), (3.69):

$$-\lambda\lambda_t = -\frac{\lambda_s}{\lambda} = b + O\left(\frac{1}{s\log s}\right) = \frac{\log s}{2s}e^{O(1)} = e^{-\sqrt{2|\log\lambda^2|}+O(1)}.$$

This yields the pointwise ode:

$$-e^{\sqrt{2|\log\lambda^2|}+O(1)}(\lambda^2)_t = 1$$

which integration in time yields:

$$\lambda^2(t) = (T-t)e^{-\sqrt{2|\log(T-t)|}+O(1)} \tag{3.72}$$

and (1.4) is proved.

<u>case $k \geq 1$</u>: We estimate from (3.21), (3.8):

$$-\frac{\lambda_s}{\lambda} = a = \frac{k+1}{2ks} - \frac{k+1}{2k^2 s\log s} + O\left(\frac{1}{s(\log s)^{3/2}}\right) \tag{3.73}$$

and hence there exists $c^* = c^*(u_0)$ such that

$$-\log\lambda(s) = \frac{k+1}{2k}\log s - \frac{k+1}{2k^2}\log\log s + c^* + o_{s\to+\infty}(1)$$

or equivalently:

$$\lambda(s) = c(u_0)(1+o(1))\frac{(\log s)^{\frac{k+1}{2k^2}}}{s^{\frac{k+1}{2k}}}, \quad c(u_0) > 0. \tag{3.74}$$

We conclude that

$$T = \int_0^{+\infty}\lambda^2(s)ds < +\infty$$

and

$$T - t = \int_s^{+\infty}\lambda^2(\sigma)\,d\sigma = \int_s^{+\infty}(c^2+o(1))\frac{(\log\sigma)^{\frac{k+1}{k^2}}}{\sigma^{\frac{k+1}{k}}}d\sigma = (kc^2+o(1))\frac{(\log s)^{\frac{k+1}{k^2}}}{s^{\frac{1}{k}}}.$$

This implies

$$\frac{1}{s} = \frac{(T-t)^k}{|\log(T-t)|^{\frac{k+1}{k}}}(c+o(1)) \tag{3.75}$$

which together with (3.74) yields the melting law:

$$\lambda(t) = (c^*(u_0) + o_{t\to T}(1))\frac{(T-t)^{\frac{k+1}{2}}}{|\log(T-t)|^{\frac{k+1}{2k}}},$$

this is (1.5).

step 2 Non concentration of the energy. Pick $R > 0$ and a cut-off function

$$\chi_R(x) = \chi\left(\frac{x}{R}\right) = \begin{cases} 0 & \text{for } \ x \leq R \\ 1 & \text{for } \ x \geq 2R. \end{cases}$$

Then for t sufficiently close to the melting time T:

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int \chi_R|\nabla u|^2\,dx = \int \chi_R \nabla u\cdot\nabla\partial_t u\,dx = \int \chi_R\nabla u\cdot\nabla\Delta u\,dx \\ = & \ -\int \Delta u\left[\chi_R\Delta u + \nabla\chi_R\cdot\nabla u\right]\,dx \\ = & \ -\int \chi_R|\Delta u|^2 + \frac{1}{2}\int|\nabla u|^2 r\frac{\partial}{\partial r}\left(\frac{\chi_R'}{r}\right)\,dx \end{aligned} \tag{3.76}$$

and hence the uniform bound on the Dirichet energy ensures:

$$\forall R > 0, \quad \int_0^T \chi_R|\Delta u|^2 dx < +\infty.$$

Hence for all $0 < \tau < T - t$,

$$\begin{aligned} & \int \chi_R|\nabla u(t+\tau) - \nabla u(t)|^2 dx = \int \chi_R\left|\int_t^{t+\tau}(\partial_t\nabla u)(\sigma,x)d\sigma\right|^2 dx \\ \lesssim & \ \tau\int_0^T \chi_R|\partial_t\nabla u|^2 dx \lesssim \tau\int_0^T \chi_R|\nabla\Delta u|^2 dx \leq C(R)\tau, \end{aligned}$$

where the last estimate follows by integrating-in-t equation (3.65) with $\chi = \chi_R$ and using (3.76). Hence for all $R > 0$, $\nabla u(t,x)$ is a Cauchy sequence in $L^2(|x| \geq 2R)$ as $t \to T$. We conclude from a simple diagonal extraction argument that there exists $u^* \in \dot{H}^1(\mathbb{R}^2)$ such that

$$\forall R > 0, \quad \nabla u(t) \to \nabla u^* \ \text{ in } \ L^2(|x| \geq 2R) \text{ as } \ t \to T. \tag{3.77}$$

Moreover, the uniform bound on the Dirichlet energy (1.3) ensures

$$\nabla u^* \in L^2, \quad \nabla u(t) \rightharpoonup \nabla u^* \quad \text{weakly in } \ L^2 \ \text{ as } \ t \to T. \tag{3.78}$$

Pick now

$$R(t) = \begin{cases} \lambda(t)B(t), & B^2(t)b(t) = \frac{1}{2}|\log b(t)| \ \text{ for } \ k = 0 \\ \lambda(t)B(t), & B^2(t)a(t) = \lambda(t)|\log a(t)| \ \text{ for } \ k \geq 1. \end{cases} \tag{3.79}$$

Note that in both cases we have $B(t) \gg 1$. Then from (3.76):

$$\left|\frac{d}{d\tau}\int \chi_{R(t)}|\nabla u(\tau)|^2 dx\right| \lesssim \frac{1}{R^2(t)}\int|\nabla u(\tau)|^2\,dx + \int \chi_{R(t)}|\Delta u(\tau)|^2\,dx$$

and hence integrating over $[t,T)$ and using (3.77), (3.79), (1.3):

$$\left|\int \chi_{R(t)}|\nabla u(\tau)|^2 dx - \int \chi_{R(t)}|\nabla u^*|^2 dx\right| \lesssim \frac{T-t}{R^2(t)} + \int_t^T\int_{r\geq\lambda(t)}|\Delta u(\tau)|^2 dx d\tau.$$

If $k = 0$, we use (1.4) to estimate

$$\frac{T-t}{R^2(t)} = \frac{2b(t)(T-t)}{\lambda^2(t)}\frac{1}{|\log b(t)|} \longrightarrow 0 \ \text{ as } \ t \to T.$$

The above limit holds since by (3.72)

$$\frac{2b(t)(T-t)}{\lambda^2(t)}\frac{1}{|\log b(t)|}\lesssim\frac{b(t)e^{\sqrt{2|\log(T-t)|}}}{|\log b(t)|}\lesssim\frac{b(t)e^{2\sqrt{|\lambda(t)|}}}{|\log b(t)|}\lesssim\frac{1}{|\log b(t)|}\to 0,\quad \text{as}\, t\to T,$$

where the last bound follows from (3.68) and (3.71). If $k\geq 1$, then

$$\frac{T-t}{R^2(t)}=\frac{a(t)(T-t)}{\lambda^2(t)}\frac{1}{c|\log a(t)|}\longrightarrow 0 \ \text{ as } \ t\to T.$$

The above limit holds since by (3.72)

$$\frac{a(t)(T-t)}{\lambda^2(t)}\frac{1}{|\log a(t)|}\lesssim\frac{\frac{(T-t)^k}{|\log(T-t)|^{\frac{k+1}{k}}}(T-t)}{\frac{(T-t)^{k+1}}{|\log(T-t)|^{\frac{k+1}{k}}}|\log a(t)|}\lesssim\frac{1}{|\log a(t)|}\to 0,\quad \text{as}\ t\to T,$$

where we used (3.73) and (3.75). Letting $t\to T$, we conclude:

$$\int|\nabla u^*|^2=\lim_{t\to T}\int\chi_{R(t)}|\nabla u(\tau)|^2dx. \tag{3.80}$$

We now claim that

$$\lim_{t\to T}\int_{\{|x|\geq\lambda(t)\}}(1-\chi_{R(t)})|\nabla u(\tau)|^2dx=0 \tag{3.81}$$

which together with (3.80), (3.78) concludes the proof of (1.6). Indeed, in the case $k=0$, from (2.78), (A.10), (3.21), and (3.79) we obtain:

$$\begin{aligned}
&\int_{\{|x|\geq\lambda(t)\}}(1-\chi_{R(t)})|\nabla u(t)|^2dx\leq\int_{\lambda(t)\leq|x|\leq 2R(t)}|\nabla u(t)|^2dx\\
&=\int_{1\leq|y|\leq 2B(t)}|\nabla v(t,y)|^2dy\\
&\lesssim e^{2bB^2(t)}\left[\int_{1\leq y\leq 2B(t)}|b\partial_y\eta_{b_0}|^2\rho_b y dy+\|\partial_y\varepsilon\|_b^2\right]\\
&=e^{2bB^2(t)}\left[b^2|\log b|^2+\frac{b^2}{|\log b|}\right]\lesssim b|\log b|^2\to\ \ 0\ \text{ as }\ t\to T,
\end{aligned}$$

and (3.81) is proved. A similar algebra holds for $k\geq 1$. This concludes the proof of Theorem 1.1.

4. **Infinite time freezing regimes**

This section is devoted to the existence and stability of the freezing process emerging from strongly localized initial data. Throughout the section, we let

$$\pm=+,\quad \rho=\rho_+,\quad B>0.$$

4.1. **Renormalized equations and initialization.** We let

$$u(t,x)=v(s,y),\quad y=\frac{r}{\lambda(t)},\quad \lambda(0)=1, \tag{4.1}$$

with the renormalized time

$$s(t)=s_0+\int_0^t\frac{d\tau}{\lambda^2(\tau)},\quad s_0\gg 1, \tag{4.2}$$

and obtain the renormalized equation:

$$\begin{cases} \partial_s v + \mathcal{H}_A v = 0, & A = \frac{\lambda_s}{\lambda}, \\ v(s,1) = 0, & \partial_y v(s,1) = -A. \end{cases} \tag{4.3}$$

We now prepare our initial data in the following way. We let

$$B(s) = \frac{1}{2s}, \quad B_k^e = \frac{1}{s^{k+1}(\log s)^2} \tag{4.4}$$

so that with $A^e = B^e$:

$$(B_k^e)_s + B_k^e B\left(2k+2+\frac{2}{\log s}\right) + \frac{2(B-A^e)B_k^e}{\log s} = O\left(\frac{1}{s^{k+2}(\log s)^3}\right). \tag{4.5}$$

We define

$$Q_\beta(y) := -\sum_{j=0}^{k} B_j \hat{\eta}_{B,j}(y) \tag{4.6}$$

and introduce the dynamical decomposition

$$v(s,y) = Q_{\beta(s)} + \varepsilon(s,y), \quad (\hat{\eta}_{B,j}(s), \varepsilon)_B = 0, \quad 0 \le j \le k. \tag{4.7}$$

We let again

$$\varepsilon_2 = \mathcal{H}_B \varepsilon, \quad \mathcal{E} := \|\mathcal{H}_B \varepsilon\|_B^2,$$

which due to the orthogonality conditions (3.14) is a coercive norm, see Appendix A. We assume the initial smallness

$$\mathcal{E}(0) \le \frac{B(B_k^e)^2(0)}{|\log B(0)|} \tag{4.8}$$

and consider the bootstrap bound

$$\mathcal{E} \le \frac{DB(B_k^e)^2}{|\log B|} \tag{4.9}$$

for some large enough universal $D = D(k)$ to be chosen later. Moreover, we assume initially

$$B_k(s_0) = B_k^e(s_0), \quad s_0 \gg 1 \tag{4.10}$$

and bootstrap the bound

$$|B_k(s)| \le 10 B_k^e(s). \tag{4.11}$$

For $k \ge 1$, we also let

$$B_j(s) = \frac{V_j(s)}{s^{k+1}(\log s)^{\frac{5}{2}}}, \quad j = 0, \dots, k-1. \tag{4.12}$$

and assume

$$\sum_{j=0}^{k-1} |V_j(s)|^2 \le K^2. \tag{4.13}$$

We define

$$s^* = \begin{cases} \sup_{s \ge s_0} \{(4.9), (4.11) \text{hold on } [s_0, s]\} & \text{for } k = 0, \\ \sup_{s \ge s_0} \{(4.9), (4.11), (4.13) \text{ hold on } [s_0, s]\} & \text{for } k \ge 1. \end{cases}$$

The main ingredient of the proof of Theorem 1.2 is the following:

Proposition 4.1 (Bootstrap estimates on B and ε)**.** *The following statements hold:*
1. Stable regime*: for $k = 0$, $s^* = +\infty$.*
2. Unstable regime*: for $k \ge 1$, there exist constants K, $\delta = \delta(K) \ll 1$ and $(V_0(0), \dots, V_{k-2}(0), V_{k-1}(0))$ depending on $\varepsilon(0)$ such that $s^* = +\infty$.*

From now on and for the rest of this section, we study the flow in the boostrap regime $s \in [s_0, s^*)$. Note in particular the rough bounds

$$|B_j| \lesssim |B_k^e|, \quad 0 \le j \le k \quad \text{and} \quad \mathcal{E} \le \frac{B(B_k^e)^2}{\sqrt{|\log B|}} \tag{4.14}$$

for $s_0 \ge s_0(K)$ large enough.

4.2. **Extraction of the leading order ODE's driving the freezing.** We start with the constraint induced by the boundary conditions:

Lemma 4.2 (Boundary conditions)**.** *There holds:*

$$A = \sum_{j=0}^{k} B_j \left[1 + \frac{2\alpha_j}{|\log B|} + O\left(\frac{1}{|\log B|^2}\right)\right] + O\left(\frac{\sqrt{\mathcal{E}}}{|\log B|\sqrt{B}}\right), \tag{4.15}$$

$$\varepsilon_2(1) = A(B-A), \tag{4.16}$$

$$\partial_y \varepsilon_2(1) = A_s - A^2(B-A) + \sum_{j=0}^{k} \hat{\lambda}_{B,j} B B_j \left[1 + \frac{2\alpha_j}{|\log B|} + O\left(\frac{1}{|\log B|^2}\right)\right] \tag{4.17}$$

Remark 4.3. Note that (4.15), (4.14) imply

$$|A| \lesssim B_k^e + \frac{\sqrt{\mathcal{E}}}{|\log B|\sqrt{B}} \lesssim B_k^e. \tag{4.18}$$

Proof of Lemma 3.5. We compute from (2.89) (3.29):

$$\partial_y \hat{\eta}_{B,j}(1) = \partial_y \eta_{B,j}(1) e^{-\frac{B}{2}} = 1 + \frac{2\alpha_j}{|\log B|} + O\left(\frac{1}{|\log B|^2}\right). \tag{4.19}$$

This implies that

$$\partial_y Q_\beta(1) = -\sum_{j=0}^{k} B_j \partial_y \eta_{B,j}(1) = -\sum_{j=0}^{k} B_j \left[1 + \frac{2\alpha_j}{|\log B|} + O\left(\frac{1}{|\log B|^2}\right)\right].$$

Since $v = Q_\beta + \varepsilon$, it follows that

$$\begin{aligned} \varepsilon_y\big|_{y=1} &= v_y\big|_{y=1} - \partial_y Q_\beta\big|_{y=1} = -\frac{\lambda_s}{\lambda} - \partial_y Q_\beta(1) \\ &= -A + \sum_{j=0}^{k} B_j \left[1 + \frac{2\alpha_j}{|\log B|} + O\left(\frac{1}{|\log B|^2}\right)\right], \end{aligned}$$

which together with (B.2) yields (4.15). From (4.3), $v(s,1) = 0$ and $\partial_y v(s,1) = -\frac{\lambda_s}{\lambda} = -A$:

$$0 = \mathcal{H}_A v(1) = (\mathcal{H}_B v + (B-A)\Lambda v)(1) = \varepsilon_2(1) + A(A-B),$$

this is (4.16). Now from $\partial_y v(s,1) = -A$, we have

$$\partial_s \partial_y v(s,1) = -A_s.$$

On the other hand, taking ∂_y of (4.3), we have:

$$0 = \partial_s \partial_y v + \partial_y(\mathcal{H}_B v + (B-A)\Lambda v) = \partial_s \partial_y v + \partial_y \varepsilon_2 + \partial_y \mathcal{H}_B Q_\beta + (B-A) y \Delta v.$$

We evaluate the above identity at $y = 1$. From (4.3) and $\partial_s v(s,1) = 0$, $\partial_y v(s,1) = -A$:

$$\Delta v(1) = -A\Lambda v(1) = A^2.$$

By construction,

$$\partial_y \mathcal{H}_b Q_\beta = -\sum_{j=0}^{k} \hat{\lambda}_{B,j} B B_j \partial_y \hat{\eta}_{B,j},$$

and hence:

$$-A_s + \partial_y \varepsilon_2(1) - \sum_{j=0}^{k} \hat{\lambda}_{B,j} B B_j \left[1 + \frac{2\alpha_j}{|\log B|} + O\left(\frac{1}{|\log B|^2}\right)\right] + A^2(B-A) = 0.$$

□

We now compute the leading order modulation equations.

Proposition 4.4 (Leading order modulation equations)**.** *Under the a priori bounds of Proposition 4.1, there holds*

$$\partial_s Q_\beta + \mathcal{H}_A Q_\beta = \mathrm{Mod} + \Psi \tag{4.20}$$

where we defined the modulation vector

$$\mathrm{Mod} \quad := \quad -\sum_{j=0}^{k} \left[(B_j)_s + B B_j \hat{\lambda}_{B,j} + \frac{2(B-A)B_j}{|\log B|}\right] \hat{\eta}_{B,j}, \tag{4.21}$$

and the remaining error satisfies the bound:

$$\|\Psi\|_b + \frac{1}{\sqrt{B}} \|\partial_y \Psi\|_B + \frac{1}{B} \|\mathcal{H}_B \Psi\|_B \lesssim \frac{\sqrt{B} B_k^e}{|\log B|}. \tag{4.22}$$

Proof of Proposition 3.6. Let the deviation

$$\Phi := B_s + 2B(B-A).$$

By definition

$$\mathcal{H}_A = \mathcal{H}_B + (B-A)\Lambda$$

and we therefore compute from (4.6):

$$\begin{aligned}
& -\partial_s Q_{\beta(s)}(y) - \mathcal{H}_A Q_\beta \\
= \quad & \sum_{j=0}^{k} \left[(B_j)_s \hat{\eta}_{B,j} + B_s \frac{B_j}{B} B \partial_B \hat{\eta}_{B,j} + B B_j \hat{\lambda}_{B,j} \hat{\eta}_{B,j} + (B-A) B_j \Lambda \hat{\eta}_{B,j}\right] \\
= \quad & \sum_{j=0}^{k} \left\{[(B_j)_s + B B_j \hat{\lambda}_{b,j}] \hat{\eta}_{B,j} + (B-A) B_j [\Lambda \hat{\eta}_{B,j} - 2B \partial_B \hat{\eta}_{B,j}] + \frac{B_j}{B} B \partial_B \hat{\eta}_{B,j} \Phi\right\} \\
= \quad & \sum_{j=0}^{k} \left\{\left[(B_j)_s + B B_j \hat{\lambda}_{B,j} + \frac{2(B-A)B_j}{|\log B|}\right] \hat{\eta}_{B,j}\right. \\
& \left. + (B-A) B_j \left(\Lambda \hat{\eta}_{B,j} - 2B \partial_B \hat{\eta}_{B,j} - \frac{2}{|\log B|} \hat{\eta}_{B,j}\right) + \frac{B_j}{B} \Phi B \partial_B \hat{\eta}_{B,j}\right\}.
\end{aligned}$$

We now estimate from (4.18):

$$|\Phi| \lesssim |AB| \lesssim B B_k^e \tag{4.23}$$

and hence using (4.14), (2.93):

$$\|\frac{B_j}{B} \Phi B \partial_B \hat{\eta}_{B,j}\|_B \lesssim \frac{|\log B|}{\sqrt{B}} (B_k^e)^2 \lesssim \frac{\sqrt{B} B_k^e}{|\log B|}$$

and similarily for higher derivatives. Moreover from (2.95), (4.14):

$$\left\|(B-A)B_j\left[\Lambda\hat{\eta}_{B,j}-2B\partial_B\hat{\eta}_{B,j}-\frac{2}{|\log B|}\hat{\eta}_{B,j}\right]\right\|_B\lesssim\frac{1}{\sqrt{B}|\log B|}BB_k^e\lesssim\frac{\sqrt{B}B_k^e}{|\log B|}$$

and similarily for higher derivatives, and (4.22) is proved. $\square$

4.3. **Modulation equations.** The relations (4.3), (4.20) yield

$$\partial_s\varepsilon+\mathcal{H}_A\varepsilon=\mathcal{F},\quad \mathcal{F}=-\text{Mod}-\Psi \tag{4.24}$$

and we now compute the exact modulation equations.

Lemma 4.5 (Modulation equations for B_j)**.** *There holds for $0\leq j\leq k$:*

$$\left|(B_j)_s+B_jB\left(2j+2+\frac{4}{|\log B|}\right)\right|\lesssim\frac{BB_k^e}{|\log B|^2}. \tag{4.25}$$

Proof of Lemma 4.5. Let $0\leq j\leq k$ and take the scalar product of (4.24) with $\hat{\eta}_{B,j}$ and use the orthogonality condition (4.7) to compute:

$$-(\varepsilon,\partial_s\hat{\eta}_{B,j})_B-B_s(\varepsilon,\frac{|y|^2}{2}\hat{\eta}_{B,j})_B=(\mathcal{F},\hat{\eta}_{B,j})+(A-B)(\Lambda\varepsilon,\hat{\eta}_{B,j}).$$

We integrate by parts unsing (4.7):

$$\begin{aligned}
&-(\varepsilon,\partial_s\hat{\eta}_{B,j})_B-B_s(\varepsilon,\frac{|y|^2}{2}\hat{\eta}_{B,j})_B+(B-A)(\Lambda\varepsilon,\hat{\eta}_{B,j})_B\\
=\;&-(\varepsilon,\partial_s\hat{\eta}_{B,j})_B-B_s(\varepsilon,\frac{|y|^2}{2}\hat{\eta}_{B,j})_B-(B-A)(\varepsilon,B|y|^2\hat{\eta}_{B,j}+\Lambda\hat{\eta}_{B,j})_B\\
=\;&-(\varepsilon,\frac{1}{2}(B_s+2B(B-A))|y|^2\hat{\eta}_{B,j})_B-(\varepsilon,\frac{B_s+2B(B-A)}{B}B\partial_B\hat{\eta}_{B,j})_B\\
&+(B-A)(\varepsilon,2B\partial_B\hat{\eta}_{B,j}-\Lambda\hat{\eta}_{B,j})_B\\
=\;&-(\varepsilon,\frac{\Phi}{B}[\frac{1}{2}B|y|^2\hat{\eta}_{B,j}+B\partial_B\hat{\eta}_{B,j}])_B+(B-A)(\varepsilon,2B\partial_B\hat{\eta}_{B,j}-\Lambda\hat{\eta}_{B,j})_B.
\end{aligned}$$

We now estimate from (B.2), (2.92), (2.93), (4.23):

$$\left|-(\varepsilon,\frac{\Phi}{B}[\frac{1}{2}B|y|^2\hat{\eta}_{B,j}+B\partial_B\hat{\eta}_{B,j}])_B\right|\lesssim\|\varepsilon\|_B\frac{|\Phi|}{B}\frac{|\log B|}{\sqrt{B}}\lesssim\frac{\sqrt{\mathcal{E}}}{|\log B|\sqrt{B}}$$

and using (2.95) and (4.7):

$$|(B-A)(\varepsilon,2B\partial_B\hat{\eta}_{B,j}-\Lambda\hat{\eta}_{B,j})_B|\lesssim\frac{B}{\sqrt{B}}\|\varepsilon\|_B\lesssim\frac{\sqrt{\mathcal{E}}}{|\log B|\sqrt{B}}.$$

We now estimate the $\mathcal{F}$ terms given by (4.24). From (4.22):

$$|(\Psi,\hat{\eta}_{B,j})_B|\lesssim\|\Psi\|_B\|\hat{\eta}_{B,j}\|_B\lesssim\frac{\sqrt{B}B_k^e}{|\log B|}\frac{|\log B|}{\sqrt{B}}\lesssim B_k^e.$$

The collection of above bounds together with (2.96) and (4.24) yields

$$\frac{|(\text{Mod},\hat{\eta}_{\text{B,j}})_{\text{B}}|}{(\hat{\eta}_{B,j},\hat{\eta}_{B,j})_B}\lesssim\frac{B}{|\log B|^2}\left[\frac{\sqrt{\mathcal{E}}}{|\log B|\sqrt{B}}+B_k^e\right]=\frac{BB_k^e}{|\log B|^2}$$

or equivalently:

$$\sum_{j=0}^{k}\left|(B_j)_s+BB_j\hat{\lambda}_{B,j}+\frac{2(B-A)B_j}{|\log B|}\right|\lesssim\frac{BB_k^e}{|\log B|^2}. \tag{4.26}$$

We conclude from (4.26), (4.18), (2.89):

$$\left|(B_j)_s + B_jB\left[2j+2+\frac{4}{|\log B|}+O\left(\frac{1}{|\log B|^2}\right)\right]\right| \lesssim \frac{BB_k^e}{|\log B|^2},$$

this is (4.25). $\square$

4.4. **Energy bound.** We now derive the energy estimate in the freezing regime

Proposition 4.6 (Energy bound for freezing)**.** *There holds the pointwise control*

$$\frac{1}{2}\frac{d}{ds}\left\{\mathcal{E}+O\left(\frac{B(B_k^e)^2}{|\log B|}\right)\right\}+B(2k+4+c)\|\varepsilon_2\|_B^2 \lesssim B\frac{B(B_k^e)^2}{|\log B|} \tag{4.27}$$

for some universal constant $c>0$.

Proof of Proposition 4.6. We compute the energy identity for $\mathcal{E}$ and estimate all terms.

step 1 Algebraic energy identity. Recall $\varepsilon_2 = \mathcal{H}_B\varepsilon$. We compute from (4.24):

$$\partial_s\varepsilon_2 + \mathcal{H}_A\varepsilon_2 = [\partial_s, \mathcal{H}_B]\varepsilon + [\mathcal{H}_A, \mathcal{H}_B]\varepsilon + \mathcal{H}_B\mathcal{F}.$$

and hence using (3.49):

$$\begin{aligned}
&[\partial_s, \mathcal{H}_B]\varepsilon + [\mathcal{H}_A, \mathcal{H}_B]\varepsilon = -B_s\Lambda\varepsilon + [\mathcal{H}_B + (B-A)\Lambda, \mathcal{H}_B]\varepsilon \\
&= -B_s\Lambda\varepsilon + (B-A)[\Lambda, -\Delta] = -B_s\Lambda\varepsilon + 2(B-A)\Delta\varepsilon \\
&= -(B_s + 2B(B-A))\Lambda\varepsilon - 2(B-A)[-\Delta\varepsilon - B\Lambda\varepsilon] = -\Phi\Lambda\varepsilon - 2(B-A)\varepsilon_2.
\end{aligned}$$

Hence the ε_2 equation:

$$\partial_s\varepsilon_2 + \mathcal{H}_A\varepsilon_2 = -\Phi\Lambda\varepsilon - 2(B-A)\varepsilon_2 + \mathcal{H}_B\mathcal{F}. \tag{4.28}$$

We now compute the modified energy identity:

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\mathcal{E} &= \frac{1}{2}\frac{d}{ds}\int_{y\geq 1}\varepsilon_2^2 e^{\frac{By^2}{2}}ydy = \frac{B_s}{4}\int_{y\geq 1}y^2|\varepsilon_2|^2e^{\frac{B|y|^2}{2}}ydy + (\partial_s\varepsilon_2, \varepsilon_2)_B \\
&= \frac{B_s}{4}\|y\varepsilon_2\|_B^2 + (-\Phi\Lambda\varepsilon - 2(B-A)\varepsilon_2 + \mathcal{H}_B\mathcal{F} - \mathcal{H}_A\varepsilon_2, \varepsilon_2)_B.
\end{aligned}$$

We carefully integrate by parts to compute:

$$\begin{aligned}
&-\int_{y\geq 1}\varepsilon_2\mathcal{H}_A\varepsilon_2 e^{\frac{By^2}{2}}ydy = -\int_{y\geq 1}\varepsilon_2[\mathcal{H}_B\varepsilon_2 + (B-A)\Lambda\varepsilon_2]e^{\frac{By^2}{2}}ydy \\
=\ & \int_{y\geq 1}\partial_y(\rho_B y\partial_y\varepsilon_2)\varepsilon_2 dy - (B-A)\int_{y\geq 1}\varepsilon_2 y\partial_y\varepsilon_2 e^{\frac{By^2}{2}}ydy \\
=\ & -\rho_B(1)\varepsilon_2(1)\partial_y\varepsilon_2(1) - \int_{y\geq 1}|\partial_y\varepsilon_2|^2e^{\frac{By^2}{2}}ydy \\
&+\frac{B-A}{2}\rho_B(1)\varepsilon_2^2(1) + \frac{B-A}{2}\int_{y\geq 1}\varepsilon_2^2\left[2+By^2\right]e^{\frac{By^2}{2}}ydy \\
=\ & -\|\partial_y\varepsilon_2\|_B^2 + (B-A)\|\varepsilon_2\|_2^2 + \frac{B(B-A)}{2}\|y\varepsilon_2\|_B^2 - \rho_B(1)\varepsilon_2(1)\left[\partial_y\varepsilon_2 - \frac{B-A}{2}\varepsilon_2\right](1).
\end{aligned}$$

This yields the algebraic energy identity:

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\mathcal{E} &= -\|\partial_y\varepsilon_2\|_B^2 - (B-A)\|\varepsilon_2\|_2^2 + \frac{\Phi}{4}\|y\varepsilon_2\|_B^2 - \rho_B\varepsilon_2\left[\partial_y\varepsilon_2 - \frac{B-A}{2}\varepsilon_2\right](1) \\
&- \Phi(\Lambda\varepsilon, \varepsilon_2)_B + (\mathcal{H}_B\mathcal{F}, \varepsilon_2)_B.
\end{aligned} \tag{4.29}$$

We now estimate all terms in the right-hand side of (4.29).

step 2 Nonlinear estimates. From (B.12), (4.23):

$$|\Phi||(\Lambda\varepsilon,\varepsilon_2)_B|\lesssim BB_k^e\frac{\mathcal{E}}{B}\lesssim B\frac{\mathcal{E}}{|\log B|}.$$

Moreover from (4.7), (4.21), $(\mathcal{H}_B\mathrm{Mod},\varepsilon_2)_{\mathrm{b}}=0$, and from (4.22), (4.14):

$$|(\mathcal{H}_B\Psi,\varepsilon_2)_B\lesssim B\sqrt{\mathcal{E}}\frac{\sqrt{B}B_k^e}{|\log B|}\lesssim B\frac{B(B_k^e)^2}{|\log B|}.$$

We now estimate from (B.14), (4.23), (4.16), (4.18):

$$\begin{aligned}
|\Phi|\|y\varepsilon_2\|_B^2 &\lesssim BB_k^e\left[\frac{\|\partial_y\varepsilon_2\|_B^2}{B^2}+\frac{\varepsilon_2^2(1)}{B}\right]\lesssim\frac{\|\partial_y\varepsilon_2\|_B^2}{|\log B|}+\frac{B^3A^2}{|\log B|}\\
&\lesssim \frac{\|\partial_y\varepsilon_2\|_B^2}{|\log B|}+\frac{B^3}{|\log B|}\left[(B_k^e)^2+\frac{\mathcal{E}}{B|\log B|^2}\right]\\
&\lesssim \frac{\|\partial_y\varepsilon_2\|_B^2}{|\log B|}+B\frac{B(B_k^e)^2}{|\log B|^2}.
\end{aligned}$$

step 3 Boundary term and conclusion. It now remains to treat the boundary term in (4.29). First from (4.16), (4.17), (4.18):

$$|B-A||\varepsilon_2(1)|^2\lesssim|B|(AB)^2\lesssim B^3(B_k^e)^2.$$

Let

$$\tilde{A}=\sum_{j=0}^k B_j\left(1+\frac{2\alpha_j}{|\log B|}\right),$$

and observe the bounds from (4.15), (4.14), (4.25):

$$|\tilde{A}-A|\lesssim\frac{B_k^e}{|\log B|},\quad |\tilde{A}_s|\lesssim BB_k^e \tag{4.30}$$

We now rewrite (4.17) using (4.25), (4.18), (4.26):

$$\begin{aligned}
\partial_y\varepsilon_2(1) &= \left(A-\tilde{A}\right)_s+\sum_{j=0}^k\left(1+\frac{2\alpha_j}{|\log B|}\right)\left[(B_j)_s+\hat{\lambda}_{B,j}BB_j\right]+O\left(\frac{BB_k^e}{|\log B|^2}\right)\\
&= (A-\tilde{A})_s+O\left(\frac{BB_k^e}{|\log B|}\right)
\end{aligned}$$

from which:

$$\begin{aligned}
&\rho_B(1)\varepsilon_2(1)\left[\partial_y\varepsilon_2-\frac{B-A}{2}\varepsilon_2\right](1)\\
&= \rho_B(1)A(B-A)\left[(A-\tilde{A})_s+O\left(\frac{BB_k^e}{|\log B|}+B^2B_k^e\right)\right]\\
&= O\left(B\left(\frac{B(B_k^e)^2}{|\log B|}\right)\right)+\rho_B(1)A(B-A)(A-\tilde{A})_s.
\end{aligned}$$

We now compute using (4.30):

$$-\rho_B(1)A(B-A)(A-\tilde{A})_s=\rho_B(1)(A-\tilde{A})_s\left[(A-\tilde{A})^2+(A-\tilde{A})(\tilde{A}-B)-\tilde{A}(B-A)\right]$$
$$=-\frac{d}{ds}\left\{\rho_B(1)\left[\frac{(A-\tilde{A})^3}{3}+\frac{(A-\tilde{A})^2(\tilde{A}-B)}{2}-(A-\tilde{A})red\tilde{A}(B-A)\right]\right\}$$
$$-\frac{B_s}{2}\rho_B(1)\left[\frac{(A-\tilde{A})^3}{3}+\frac{(A-\tilde{A})^2(\tilde{A}-B)}{2}-(A-\tilde{A})\tilde{A}(B-A)\right]$$
$$+\rho_B(1)\left[\frac{(A-\tilde{A})^2}{2}(\tilde{A}_s-B_s)-(A-\tilde{A})(\tilde{A}_s(B-A)+(B_s-A_s)\tilde{A})\right]$$
$$=\quad\frac{d}{ds}\left\{O\left(\frac{B(B_e^k)^2}{|\log B|^2}\right)\right\}+O\left(\frac{B^2(B_k^e)^2}{|\log B|}\right)$$

Injecting the collection of above bounds into (4.29) yields:

$$\frac{1}{2}\frac{d}{ds}\left\{\mathcal{E}+O\left(\frac{B(B_k^2)^2}{|\log B|}\right)\right\}$$
$$=\quad-\|\partial_y\varepsilon_2\|_B^2-(B-A)\|\varepsilon_2\|_B^2+O\left(\frac{\|\partial_y\varepsilon_2\|_B^2}{|\log B|}+B\frac{B(B_k^e)^2}{|\log B|}\right)$$

and hence using the coercivity (B.13), (4.16), (4.4) and (4.18):

$$\frac{1}{2}\frac{d}{ds}\left\{\mathcal{E}+O\left(\frac{B(B_k^2)^2}{|\log B|}\right)\right\}+B\left[2k+5+O\left(\frac{1}{|\log B|}\right)\right]\|\varepsilon_2\|_B^2\lesssim\frac{B(B_k^e)^2}{|\log B|},$$

and (4.27) is proved. □

4.5. **Proof of Proposition 4.1 and Theorem 1.2.** We may now close the bootstrap estimates of Proposition 4.1.

step 1 Closing the energy bound. First observe $\left|\frac{\lambda_s}{\lambda}\right|=|A|\lesssim B$ and thus $|\log\lambda(s)|\lesssim C(s)$ implies $\lambda(s)>0$ on $[0,s^*]$. The control of the $\mathcal{E}$ norm easily implies the H^2 control $\|u(s)\|_{H^2}\lesssim C(s)$ and hence the solution is well defined from the point of view of the H^2 Cauchy theory on $[0,s^*]$. We now integrate in time the bound (4.27) and obtain for some $c>0$

$$\begin{aligned}\mathcal{E}(s)\quad&\leq\quad\left(\frac{s_0}{s}\right)^{2k+3+c}\mathcal{E}(0)+C\frac{B(s)(B_k^e(s))^2}{|\log s|}+C\frac{1}{s^{2k+3+c}}\int_{s_0}^s\frac{B^2(B_k^e)^2}{|\log B|}\sigma^{2k+3+c}\,d\sigma\\&\leq\quad\left(\frac{s_0}{s}\right)^{2k+3+c}\mathcal{E}(0)+C\frac{B(s)(B_k^e(s))^2}{|\log s|}\lesssim\frac{B(s)(B_k^e(s))^2}{|\log s|}.\end{aligned}$$

Hereby we used the explicit formulas (4.4) to infer that $\frac{1}{s^{2k+3+c}}\int_0^s\frac{B^2(B_k^e)^2}{|\log B|}\sigma^{2k+3+c}\,d\sigma\lesssim\frac{B(s)(B_k^e(s))^2}{|\log s|}$ and in the last inequality the initial data assumption (4.8). This closes the energy bound (4.9).

step 2 Control of B_j. We estimate from (4.25):

$$\left|(B_k)_s+BB_k\left(2k+2+\frac{4}{|\log B|}\right)\right|\lesssim\frac{BB_k^e}{|\log B|^2}$$

from which

$$\left|\frac{d}{ds}(s^{k+1}(\log s)^2B_k)\right|\lesssim\frac{1}{s(\log s)^2}.$$

An integration-in-time yields:

$$B_k(s) = \frac{1}{s^{k+1}(\log s)^2}\left[\frac{B_k(s_0)}{B_k^e(s_0)} + O\left(\frac{1}{\log s_0}\right)\right] \tag{4.31}$$

and the initial data assumption (4.10) now improves (4.11).
For $k \geq 1$, we now argue by contradiction and assume that for all $(V_j)_{0\leq j\leq k-1}$ with $\sum_{j=0}^{k-1}|V_j(0)|^2 \leq K^2$, there holds $s^* < +\infty$ i.e.

$$\sum_{j=0}^{k-1}|V_j(s^*)|^2 = K^2.$$

We estimate using the variables (4.12) and (4.25):

$$\left|(V_j)_s - \frac{k-j}{s}V_j\right| \lesssim \frac{1}{s\sqrt{\log s}}, \quad j = 1, \dots, k-1.$$

Hence at the exit time:

$$\frac{1}{2}\frac{d}{ds}\left[\sum_{j=0}^{k-1}|V_j|^2\right](s^*) \gtrsim \frac{1}{s^*}\sum_{j=0}^{k-1}|V_j(s^*)|^2 > 0$$

and a contradiction follows as in the melting case using Brouwer fixed point theorem. This concludes the proof of Proposition 4.1.

4.6. **Proof of Theorem 1.2.** The proof of Theorem 1.2 now follows from a simple time integration of the modulation equations.
We estimate from (4.25)

$$\left|(B_k)_s + \frac{B_k}{s}\left(k+1+\frac{2}{\log s}\right)\right| \lesssim \frac{B_k^e}{s(\log s)^2}$$

and hence

$$\left|\frac{d}{ds}(s^{k+1}(\log s)^2 B_k)\right| \lesssim \frac{1}{s(\log s)^2}$$

which implies for s large enough:

$$B_k(s) = \frac{c(u_0)(1+o(1))}{s^{k+1}(\log s)^2}$$

for some universal constant $c = c(u_0)$. We conclude from (4.15):

$$\frac{\lambda_s}{\lambda} = A = \frac{c(u_0)(1+o(1))}{s^{k+1}(\log s)^2}$$

from which there exists $\lambda_\infty \geq \lambda_\infty(u_0) > 0$ with

$$\lambda_\infty - \lambda(s) = \begin{cases} \frac{c(u_0)(1+o_{s\to+\infty}(1))}{\log s} & \text{if } k = 0, \\ \frac{c(u_0)(1+o_{s\to+\infty}(1))}{s^k(\log s)^2} & \text{if } k \geq 1. \end{cases}$$

Since for large $s \gg 1$ we have $\frac{ds}{dt} \sim \frac{1}{\lambda_\infty^2}$, (1.8) and (1.9) follow. Finally, recalling that $\hat{\eta}_{B,j} = e^{-\frac{By^2}{2}}\eta_{B,j}$ we estimate:

$$\begin{aligned}\int_{|y|\geq 1}|\nabla(\hat{\eta}_{B,j})|^2 &= \int_{y\geq 1}|\partial_y\eta_{B,j} - By\eta_{B,j}|^2\, e^{-By^2}y\,dy = \int_{z\geq\sqrt{B}}|\partial_z\psi_{B,j} - z\psi_{B,j}|^2\, e^{-z^2}z\,dz \\ &= |\log B|^2[\frac{1}{4}+o(1)]\int_{z\geq 0}|P_j' - zP_j|^2 e^{-z^2}z\,dz = c_k\,[1+o(1)]\,|\log B|^2\end{aligned}$$

for some universal constant $c_k>0$. Note that we used (2.27). Hence using (B.2):

$$\int |\nabla(v-B_k\hat{\eta}_{B,k})|^2 \lesssim |\log B|^2\sum_{j=0}^{k-1}B_j^2+\|\nabla\varepsilon\|_{L^2}^2\lesssim (B_k^e)^2.$$

Therefore, since the self-similar rescaling preserves the Dirichlet energy, it follows that

$$\int_{\Omega(t)}|\nabla u|^2=\int_{|y|\geq 1}|\nabla v|^2=B_k^2\int_{|y|\geq 1}|\nabla\hat{\eta}_{B,k}|^2+O((B_k^e)^2)$$

which yields (1.10). To prove (1.7), note that by integrating (1.1) and using the Stokes theorem, we arrive at the following conservation law:

$$\frac{d}{dt}\left(\int_{\Omega(t)}u(t,x)\,dx-\pi\lambda^2(t)\right)=0, \tag{4.32}$$

which holds as long as $u\in L^1(\Omega(t))$. To see this and evaluate $\lim_{t\to\infty}\|u\|_{L^1(\Omega(t))}$ first observe that v satisfies

$$\|v\|_{L^1(\Omega)}=\int_{y\geq 1}|v|\,ydy\leq \|v\|_B\left(\int_{y\geq 1}e^{-\frac{By^2}{2}}\,ydy\right)^{1/2}\lesssim\frac{1}{\sqrt{B}}\|v\|_B.$$

On the other hand by (4.7) it follows that

$$\|v\|_B\leq\sum_{j=0}^{k}|B_j|\|\eta_{B,j}\|_B+\|\varepsilon\|_B\lesssim\frac{B}{|\log B|^2}\frac{|\log B|}{\sqrt{B}}+\frac{1}{B}\|\mathcal{H}_B\varepsilon\|_B,\lesssim\frac{\sqrt{B}}{|\log B|}$$

where we used (4.4), (2.96), (4.12), (B.12), and (4.9). The two previous inequalities lead to

$$\|v\|_{L^1(\Omega)}\lesssim\frac{1}{|\log B|}\to 0\quad\text{as } s\to\infty.$$

Therefore

$$\|u\|_{L^1(\Omega(t))}=\lambda^2(t)\|v\|_{L^1(\Omega)}\to 0\quad\text{as } t\to\infty$$

since $|\lambda(t)|$ remains bounded. It follows in particular that the conservation law (4.32) holds and formula (1.7) follows.
This concludes the proof of Theorem 1.2.

Appendix A. Coercivity estimates in the melting case

This appendix is devoted to the derivation of various coercivity estimates in the melting regime

$$\pm=-,\quad \rho=\rho_-,\quad b>0$$

which are used along the proof. We start with the standard compactness of the harmonic oscillator.

Lemma A.1 (Weighted L^2 estimate)**.** *Let $u,\partial_z u\in L^2_\rho(\mathbb{R}^2)$. Then $\forall k\geq 0$,*

$$\int z^{2k}u^2z\rho dz\lesssim_k\int(\partial_z u)^2\rho zdz+\int u^2z\rho dz. \tag{A.1}$$

Proof of Lemma A.1. Indeed, we use $\partial_z\rho=-z\rho$ and integrate by parts to compute:

$$\int(\partial_z u-\delta z^k u)^2\rho zdz=\int(\partial_z u)^2\rho zdz+\delta^2\int z^{2k}u^2\rho zdz-2\delta\int z^{k+1}u\partial_z u\rho zdz$$

$$=\int(\partial_z u)^2\rho zdz+\delta^2\int z^{2k}u^2\rho zdz-\delta[z^{k+1}\rho u^2]_0^{+\infty}+\delta\int u^2[(k+1)^{k-1}z\rho-z^{2k}\rho]dz$$

and hence for $0 < \delta = \delta(k) \ll 1$ small enough:

$$\int z^{2k} u^2 \rho z dz \lesssim_k \int (\partial_z u)^2 \rho z dz + \int u^2 \rho z dz + \int u^2 z^{2k-2} \rho z dz$$

and (A.1) follows by induction on $k \geq 1$. $\square$

We now claim the main coercivity property at the heart of the energy estimate.

Lemma A.2 (Coercivity of H_b)**.** *Let $k \in \mathbb{N}$ and $0 < b < b^*(k)$ small enough. Let $u \in H^3_\rho(r \geq \sqrt{b})$ satisfy*

$$u(\sqrt{b}) = 0, \quad \langle u, \psi_{b,j} \rangle_b = 0, \quad 0 \leq j \leq k.$$

Then the following inequality holds:

$$\begin{aligned} \|H_b u\|^2_{L^2_{\rho,b}} &\gtrsim \|\Delta u\|^2_{L^2_{\rho,b}} + \|(1+z)\partial_z u\|^2_{L^2_{\rho,b}} \\ &+ \|(1+z)u\|^2_{L^2_{\rho,b}} + b|\log b|^2 (\partial_z u)^2(\sqrt{b}). \end{aligned} \tag{A.2}$$

Moreover, there exists a constant $c_k > 0$ such that

$$\|\partial_z H_b u\|^2_{L^2_{\rho,b}} \geq \left[2k + 2 + O\left(\frac{1}{|\log b|}\right)\right] \|H_b u\|^2_{L^2_{\rho,b}} - c_k b^2 |H_b u(\sqrt{b})|^2 \tag{A.3}$$

and

$$\|z H_b u\|^2_{L^2_{\rho,b}} \lesssim \|\partial_z H_b u\|^2_{L^2_{\rho,b}} + b|H_b u(\sqrt{b})|^2. \tag{A.4}$$

Proof of Lemma A.2. This lemma is a simple perturbative consequence of the harmonic oscillator estimate (2.8), (2.32), and a careful integration by parts to track the boundary term in (A.2).

step 1 Proof of (A.2). Pick a small constant $\delta > 0$, then from $u(\sqrt{b}) = 0$, we may integrate by parts and compute:

$$\|(H_b - \delta)u\|^2_{L^2_{\rho,b}} = \langle (H_b - \delta)u, (H_b - \delta)u \rangle_b = \|H_b u\|^2_{L^2_{\rho,b}} - 2\delta \langle H_b u, u \rangle_b + \delta^2 \|u\|^2_{L^2_{\rho,b}}.$$

We may now use the spectral gap bound (2.32) with (A.1) and conclude that for δ small enough:

$$\|H_b u\|^2_{L^2_{\rho,b}} \gtrsim \|\partial_z u\|^2_{L^2 \rho,b} + \|(1+z)u\|^2_{L^2_{\rho,b}} + \|(H_b - \delta)u\|^2_{L^2_{\rho,b}}. \tag{A.5}$$

We integrate by parts using the general formula

$$\langle \partial_z u, v \rangle_b = -\sqrt{b} u(\sqrt{b}) v(\sqrt{b}) \rho(\sqrt{b}) - \langle u, \partial_z v \rangle_b - \langle u, \frac{v}{z} \rangle_b + \langle u, zv \rangle_b \tag{A.6}$$

to compute:

$$\begin{aligned} \|H_b u\|^2_{L^2_{\rho,b}} &= \|\Delta u\|^2_{L^2_{\rho,b}} - 2\langle u_{zz} + \frac{\partial_z u}{z}, z\partial_z u \rangle_b + \|\Lambda u\|^2_{L^2_{\rho,b}} \\ &= \|\Delta u\|^2_{L^2_{\rho,b}} - \langle \partial_z (\partial_z u)^2, z \rangle_b - 2\|\partial_z u\|^2_{L^2_{\rho,b}} + \|\Lambda u\|^2_{L^2_{\rho,b}} \\ &= \|\Delta u\|^2_{L^2_{\rho,b}} + b(\partial_z u)^2(\sqrt{b})\rho(\sqrt{b}) + \langle (\partial_z u)^2, 1 \rangle_b + \langle (\partial_z u)^2, 1 \rangle_b \\ &\quad - \|\Lambda u\|^2_{L^2_{\rho,b}} - 2\|\partial_z u\|^2_{L^2_{\rho,b}} + \|\Lambda u\|^2_{L^2_{\rho,b}} \\ &= \|\Delta u\|^2_{L^2_{\rho,b}} + b(\partial_z u)^2(\sqrt{b})\rho(\sqrt{b}). \end{aligned} \tag{A.7}$$

On the other hand, integrating by parts:

$$\langle H_b u, \log z \rangle_b = \langle u, 1 \rangle_b + \sqrt{b} \partial_z u(\sqrt{b}) \log(\sqrt{b})$$

and thus:

$$b|\partial_z u(\sqrt{b})|^2 \lesssim \frac{1}{|\mathrm{log} b|^2}\left[\|H_b u\|^2_{L^2_{\rho,b}} + \|u\|^2_{L^2_{\rho,b}}\right] \lesssim \frac{1}{|\mathrm{log} b|^2}\|H_b u\|^2_{L^2_{\rho,b}},$$

where we used (A.5). Together with (A.7), (A.5), claim (A.2) follows.

step 2 Proof of (A.3), (A.4). Define the radially symmetric function

$$v(z) = \begin{cases} H_b u(\sqrt{b}) & \text{for } 0 \le z \le \sqrt{b} \\ H_b u(z) & \text{for } z \ge \sqrt{b} \end{cases} \tag{A.8}$$

and note that $v \in H^1_{\rho,0}$. Consider

$$w := v - \sum_{j=0}^{k} \frac{\langle v, P_j\rangle_0}{\langle P_j, P_j\rangle_0} P_j.$$

Then from (2.8):

$$\int_{z\ge 0} |\partial_z w|^2 e^{-\frac{z^2}{2}} z dz \ge (2k+2)\int_{z\ge 0} |w|^2 e^{-\frac{z^2}{2}} z dz \ge (2k+2)\int_{z\ge\sqrt{b}} |w|^2 e^{-\frac{z^2}{2}} z dz. \tag{A.9}$$

On the other hand, from (2.27):

$$\left\|P_k - \frac{2}{|\mathrm{log} b|}\psi_{b,k}\right\|_{H^2_{\rho,b}} \lesssim \frac{1}{|\mathrm{log} b|}$$

from which for $0 \le j \le k$:

$$\langle v, P_j\rangle_0 = H_b u(\sqrt{b})\int_{0\le z\le\sqrt{b}} P_j e^{-\frac{z^2}{2}} z dz + \langle H_b u, \frac{2}{|\mathrm{log} b|}\psi_{b,j}\rangle_b + O\left(\frac{\|H_b u\|_{L^2_{\rho,b}}}{|\mathrm{log} b|}\right)$$

$$= \frac{2}{|\mathrm{log} b|}\langle u, \lambda_{b,j}\psi_{b,j}\rangle_b + O\left(b|H_b u(\sqrt{b})| + \frac{\|H_b u\|_{L^2_{\rho,b}}}{|\mathrm{log} b|}\right) = O\left(b|H_b u(\sqrt{b})| + \frac{\|H_b u\|_{L^2_{\rho,b}}}{|\mathrm{log} b|}\right),$$

where we used the orthogonality $\langle u, \psi_{b,j}\rangle_b = 0$, $0 \le j \le k$. Therefore

$$\|v - w\|_{H^1_{\rho,b}} \lesssim b|H_b u(\sqrt{b})| + \frac{\|v\|_{L^2_{\rho,b}}}{|\mathrm{log} b|}.$$

Injecting this into (A.9) yields (A.3). We now apply (A.1) to v and conclude from (A.3), (A.8):

$$\begin{aligned} \|zH_b u\|^2_{L^2_{\rho,b}} &\lesssim \|zv\|^2_{L^2_{\rho,0}} + b^2|H_b u(\sqrt{b})|^2 \lesssim \|\partial_z v\|^2_{L^2_{\rho,0}} + \|v\|^2_{L^2_{\rho,0}} + b^2|H_b u(\sqrt{b})|^2 \\ &\lesssim \|\partial_z H_b u\|^2_{L^2_{\rho,b}} + \|H_b u\|^2_{L^2_{\rho,b}} + b|H_b u(\sqrt{b})|^2 \lesssim \|\partial_z H u\|^2_{L^2_{\rho,b}} + b|H_b u(\sqrt{b})|^2 \end{aligned}$$

and (A.4) is proved. □

We now renormalize Lemma A.2 by letting $\varepsilon(y) = u(\sqrt{b}y)$ which yields exactly:

Lemma A.3 (Coercivity of $\mathcal{H}_b$)**.** *Let $k \in \mathbb{N}$ and $0 < b < b^*(k)$ small enough. If $\varepsilon \in H^3_{\rho_b}(y \ge 1)$ satisfies*

$$\varepsilon(1) = 0, \quad (\varepsilon, \eta_{b,j})_b = 0 \quad \textit{for} \quad 0 \le j \le k,$$

then

$$\begin{aligned} \|\mathcal{H}_b\varepsilon\|^2_b &\gtrsim \|\Delta\varepsilon\|^2_b + b\|\partial_y\varepsilon\|^2_b + b^2\|\Lambda\varepsilon\|^2_b \\ &+ b^2\|(1+\sqrt{b}y)\varepsilon\|^2_b + b|\mathrm{log} b|^2(\partial_y\varepsilon)^2(1). \end{aligned} \tag{A.10}$$

Moreover,

$$\|\partial_y \mathcal{H}_b\varepsilon\|^2_{L^2_{\rho,b}} \geq \left[2k+2+O\left(\frac{1}{|\log b|}\right)\right] b\|\mathcal{H}_b\varepsilon\|_b^2 - c_k b^2|\mathcal{H}_b\varepsilon(1)|^2 \tag{A.11}$$

and

$$b\|y\mathcal{H}_b\varepsilon\|_b^2 \lesssim \frac{1}{b}\|\partial_y\mathcal{H}_b\varepsilon\|_b^2 + |\mathcal{H}_b\varepsilon(1)|^2. \tag{A.12}$$

Appendix B. Coercivity estimates in the freezing case

We now consider the freezing regime

$$\pm = +, \quad \rho = \rho_+, \quad B > 0$$

and the operator

$$H_B = -\Delta - \Lambda, \quad u(\sqrt{B}) = 0.$$

We claim the analogue of Lemma A.2:

Lemma B.1 (Coercivity of H_B)**.** *Let $k \in \mathbb{N}$ and $0 < B < B^*(k)$ small enough. Let*

$$\hat{\psi}_{B,j} = \psi_{B,j}e^{-\frac{B|y|^2}{2}} \tag{B.1}$$

and $u \in H^3_\rho(r \geq \sqrt{B})$ satisfy

$$u(\sqrt{B}) = 0, \quad \langle u, \hat{\psi}_{B,j}\rangle_B = 0, \quad 0 \leq j \leq k,$$

then the following inequality holds:

$$\begin{aligned}\|H_B u\|^2_{L^2_{\rho,B}} &\gtrsim \|\Delta u\|^2_{L^2_{\rho,B}} + \|(1+z)\partial_z u\|^2_{L^2_{\rho,B}} \\ &+ \|(1+z)u\|^2_{L^2_{\rho,B}} + B|\log B|^2(\partial_z u)^2(\sqrt{B}).\end{aligned} \tag{B.2}$$

Moreover, there exists a constant $c_k > 0$ such that

$$\|\partial_z H_b u\|^2_{L^2_{\rho,B}} \geq \left[2k+4+O\left(\frac{1}{|\log b|}\right)\right]\|H_B u\|^2_{L^2_{\rho,B}} - c_k B^2|H_b u(\sqrt{B})|^2 \tag{B.3}$$

and

$$\|zH_B u\|^2_{L^2_{\rho,B}} \lesssim \|\partial_z H_B u\|^2_{L^2_{\rho,B}} + B|H_B u(\sqrt{B})|^2. \tag{B.4}$$

Proof. We follow the proof of Lemma (A.3).

step 1 Proof of (A.2). Pick a small constant $\delta > 0$, then from $u(\sqrt{B}) = 0$, we may integrate by parts and compute:

$$\|(H_B-\delta)u\|^2_{L^2_{\rho,b}} = \langle (H_B-\delta)u, (H_B-\delta)u\rangle_B = \|H_B u\|^2_{L^2_{\rho,B}} - 2\delta\langle H_B u, u\rangle_B + \delta^2\|u\|^2_{L^2_{\rho,B}}.$$

We now use the isometry (2.97) and (2.9). We first obtain from (2.32) the spectral gap:

$$\forall u \ \text{ with } \ \langle u, \psi_{B,j}\rangle_B = 0, \quad 0 \leq j \leq k,$$

then

$$\langle H_B u, u\rangle_B \geq \left[2k+4+O\left(\frac{1}{|\log B|}\right)\right]\|u\|^2_{L^2_{\rho,B}}, \tag{B.5}$$

and similarly from (A.1):

$$\int z^2u^2z\rho dz \lesssim \int (\partial_z u)^2\rho z dz + \int u^2 z\rho dz. \tag{B.6}$$

We therefore conclude that for δ small enough:

$$\|H_B u\|^2_{L^2_{\rho,B}} \gtrsim \|\partial_z u\|^2_{L^2\rho,B} + \|(1+z)u\|^2_{L^2_{\rho,B}}. \tag{B.7}$$

We integrate by parts using the general formula

$$\langle \partial_z u, v\rangle_B = -\sqrt{B}u(\sqrt{b})v(\sqrt{B})\rho(\sqrt{B}) - \langle u, \partial_z v\rangle_B - \langle u, \frac{v}{z}\rangle_B - \langle u, zv\rangle_B \tag{B.8}$$

to compute:

$$\begin{aligned}
\|H_B u\|^2_{L^2_{\rho,B}} &= \|\Delta u\|^2_{L^2_{\rho,B}} + 2\langle u_{zz} + \frac{\partial_z u}{z}, z\partial_z u\rangle_B + \|\Lambda u\|^2_{L^2_{\rho,B}} \\
&= \|\Delta u\|^2_{L^2_{\rho,B}} + \langle \partial_z(\partial_z u)^2, z\rangle_B + 2\|\partial_z u\|^2_{L^2_{\rho,B}} + \|\Lambda u\|^2_{L^2_{\rho,B}} \\
&= \|\Delta u\|^2_{L^2_{\rho,B}} - B(\partial_z u)^2(\sqrt{b})\rho(\sqrt{b}) - \langle(\partial_z u)^2, 1\rangle_B - \langle(\partial_z u)^2, 1\rangle_b \\
&\quad - \|\Lambda u\|^2_{L^2_{\rho,B}} + 2\|\partial_z u\|^2_{L^2_{\rho,B}} + \|\Lambda u\|^2_{L^2_{\rho,B}} \\
&= \|\Delta u\|^2_{L^2_{\rho,B}} - B(\partial_z u)^2(\sqrt{B})\rho(\sqrt{B}).
\end{aligned} \tag{B.9}$$

On the other hand, let

$$\chi(z) = \begin{cases} 1 & \text{for } \sqrt{B} \le z \le 1 \\ 0 & \text{for } z \ge 2 \end{cases}$$

then integrating by parts:

$$\langle H_B u, \chi(z)\text{log} z\rangle_B = -\langle u, H_B(\chi(z)\log z)\rangle_B + \sqrt{B}\partial_z u(\sqrt{B})\text{log}(\sqrt{B})$$

and thus:

$$B|\partial_z u(\sqrt{B})|^2 \lesssim \frac{1}{|\text{log}B|^2}\left[\|H_B u\|^2_{L^2_{\rho,B}} + \|u\|^2_{L^2_{\rho,B}}\right] \lesssim \frac{1}{|\log B|^2}\|H_B u\|^2_{L^2_{\rho,B}},$$

where we used (B.7). Together with (B.9), (B.7), claim (B.2) follows.

step 2 Proof of (B.3), (B.4). Define the radially symmetric function

$$v(z) = \begin{cases} H_B u(\sqrt{B}) & \text{for } 0 \le z \le \sqrt{B} \\ H_B u(z) & \text{for } z \ge \sqrt{B} \end{cases} \tag{B.10}$$

and note that $v \in H^1_{\rho,0}$. Consider

$$w := v - \sum_{j=0}^{k} \frac{\langle v, \hat{P}_j\rangle_0}{\langle \hat{P}_j, \hat{P}_j\rangle_0}\hat{P}_j.$$

Then from (2.11):

$$\begin{aligned}
\int_{z\ge 0} |\partial_z w|^2 e^{\frac{z^2}{2}} z dz &\ge (2k+4)\int_{z\ge 0} |w|^2 e^{\frac{z^2}{2}} z dz \\
&\ge (2k+4)\int_{z\ge\sqrt{b}} |w|^2 e^{\frac{z^2}{2}} z dz.
\end{aligned} \tag{B.11}$$

On the other hand, from (2.27), (B.1):

$$\left\|\hat{P}_k - \frac{2}{|\text{log}B|}\hat{\psi}_{B,k}\right\|_{H^2_{\rho,B}} \lesssim \frac{1}{|\text{log}B|}$$

from which for $0 \le j \le k$:

$$\begin{aligned}\langle v, \hat{P}_j\rangle_0 &= H_Bu(\sqrt{B})\int_{0\le z\le\sqrt{B}} \hat{P}_j e^{\frac{z^2}{2}} zdz + \langle H_Bu, \frac{2}{|\log B|}\hat{\psi}_{B,j}\rangle_B + O\left(\frac{\|H_Bu\|_{L^2_{\rho,B}}}{|\log B|}\right)\\ &= O\left(B|H_Bu(\sqrt{B})| + \frac{\|H_Bu\|_{L^2_{\rho,B}}}{|\log B|}\right),\end{aligned}$$

where we used the orthogonality $\langle u, \hat{\psi}_{B,j}\rangle_B = 0$, $0 \le j \le k$. Therefore

$$\|v - w\|_{H^1_{\rho,B}} \lesssim B|H_Bu(\sqrt{B})| + \frac{\|v\|_{L^2_{\rho,B}}}{|\log B|}.$$

Injecting this into (B.11) yields (B.3). We now apply (B.6) to v and conclude from (A.11), (B.10):

$$\begin{aligned}\|zH_Bu\|^2_{L^2_{\rho,B}} &\lesssim \|zv\|^2_{L^2_{\rho,0}} + B^2|H_Bu(\sqrt{B})|^2 \lesssim \|\partial_z v\|^2_{L^2_{\rho,0}} + \|v\|^2_{L^2_{\rho,0}} + B^2|H_Bu(\sqrt{B})|^2\\ &\lesssim \|\partial_z H_Bu\|^2_{L^2_{\rho,B}} + \|H_Bu\|^2_{L^2_{\rho,B}} + B|H_Bu(\sqrt{B})|^2 \lesssim \|\partial_z H_Bu\|^2_{L^2_{\rho,B}} + B|H_Bu(\sqrt{B})|^2\end{aligned}$$

and (B.4) is proved. □

In analogy to Lemma A.3, we now renormalize Lemma A.3 by letting $\varepsilon(y) = u(\sqrt{B}y)$ and obtain:

Lemma B.2 (Coercivity of $\mathcal{H}_B$). *Let $k \in \mathbb{N}$ and $0 < B < B^*(k)$ small enough. If $\varepsilon \in H^3_{\rho_B}(y \ge 1)$ satisfies*

$$\varepsilon(1) = 0, \quad (\varepsilon, \eta_{B,j})_b = 0 \ \ \textit{for} \ \ 0 \le j \le k,$$

then

$$\begin{aligned}\|\mathcal{H}_B\varepsilon\|^2_B &\gtrsim \|\Delta\varepsilon\|^2_B + b\|\partial_y\varepsilon\|^2_B + B^2\|\Lambda\varepsilon\|^2_B\\ &+ B^2\|(1+\sqrt{B}y)\varepsilon\|^2_B + B|\log B|^2(\partial_y\varepsilon)^2(1).\end{aligned} \tag{B.12}$$

Moreover,

$$\|\partial_y\mathcal{H}_B\varepsilon\|^2_{L^2_{\rho,B}} \ge \left[2k+4+O\left(\frac{1}{|\log B|}\right)\right] B\|\mathcal{H}_B\varepsilon\|^2_B - c_kB^2|\mathcal{H}_B\varepsilon(1)|^2 \tag{B.13}$$

and

$$B\|y\mathcal{H}_B\varepsilon\|^2_B \lesssim \frac{1}{B}\|\partial_y\mathcal{H}_B\varepsilon\|^2_B + |\mathcal{H}_B\varepsilon(1)|^2. \tag{B.14}$$

Appendix C. Non trivial melting initial data

In this appendix, we show that our set of initial data for melting is non empty and contains compactly supported data arbitrarily small in $\dot{H}^1$. We show the construction for $k = 0$, an analogous construction holds for $k \ge 1$ and is left to the reader.

To see this define a cut-off function $\chi(y) = 1$ for $y \le 1$ and $\chi(y) = 0$ for $y \ge 2$, and set $\chi_B = \chi(\frac{y}{B})$, where

$$B^2 = \frac{|\log b_0|}{2b_0}.$$

By abuse of notation, we denote by b_0 the initial value of b_0 in this section. Let

$$\alpha := \frac{\|\eta_{b_0}\|_{b_0^2}}{(\chi_B\eta_{b_0}, \eta_{b_0})_{b_0}}. \tag{C.1}$$

Note that $\alpha - 1 = \frac{((1-\chi_B)\eta_{b_0},\eta_{b_0})_{b_0}}{(\chi_B\eta_{b_0},\eta_{b_0})_{b_0}}$. Furthermore

$$
\begin{aligned}
&\left|((1-\chi_B)\eta_{b_0},\eta_{b_0})_{b_0}\right| \lesssim \int_{y\geq B} \eta_{b_0}^2\rho_{b_0} \lesssim \int_{y\geq B} \log^2(y)\rho_{b_0} + \|\eta_{b_0,1}\|_b^2 \\
&= \frac{1}{b_0}\int_{\sqrt{b_0}B}^{\infty} \log^2(\sqrt{b_0}z)\rho(z)\,dz + \frac{1}{b_0|\log b_0|^2} \\
&\lesssim \frac{\log^2 b_0}{b_0}\int_{\sqrt{b_0}B}^{\infty}\rho(z)\,dz + \frac{1}{b_0}\int_{\sqrt{b_0}B}^{\infty}\sqrt{\rho(z)}\,dz + \frac{1}{b_0|\log b_0|^2} \\
&\lesssim \frac{\log^2 b_0}{b_0}e^{-b_0B^2/2} + \frac{1}{b_0}e^{-b_0B^2/4} + \frac{1}{b_0|\log b_0|^2} \\
&\lesssim \frac{\log^2 b_0}{b_0}e^{-|\log b_0|/4} + \frac{1}{b_0}e^{-|\log b_0|/8} + \frac{1}{b_0|\log b_0|^2} \lesssim \frac{1}{b_0|\log b_0|^2},
\end{aligned} \tag{C.2}
$$

where we used (2.77) and (2.78). Therefore

$$
|\alpha - 1| \lesssim \frac{\frac{1}{b_0|\log b_0|^2}}{\|\eta_{b_0}\|_{b_0}^2 - ((1-\chi_B)\eta_{b_0},\eta_{b_0})_{b_0}} \lesssim \frac{\frac{1}{b_0|\log b_0|^2}}{\|\eta_{b_0}\|_{b_0}^2} \lesssim \frac{\frac{1}{b_0|\log b_0|^2}}{\frac{|\log b_0|^2}{b_0}} = \frac{1}{|\log b_0|^4}, \tag{C.3}
$$

where we used (C.2) and (2.78). Consider the initial data

$$
u(0) = v(0) = b_0\alpha\chi_B\eta_{b_0}.
$$

Note that by (C.2) we have the bound $|\alpha| \lesssim 1$. Then using (2.77), (2.78):

$$
\begin{aligned}
\int |\partial_y u(0)|^2 &\lesssim b_0^2\left[\int_{1\leq y\leq 2B} |\partial_y\eta_{b_0}|^2 + \int_{B_0\leq y\leq 2B_0}\frac{|\eta_{b_0}|^2}{B^2}\right] \\
&\lesssim b_0^2 e^{b_0B^2/2}\left[\|\partial_y\eta_{b_0}\|_{b_0}^2 + \frac{b_0}{K}\|\eta_{b_0}\|_{b_0}^2\right] \lesssim b_0^2 e^{\frac{|\log b_0|}{4}}\left[|\log b_0| + \frac{b_0}{K}\frac{|\log b_0|}{2b_0}\right] \\
&\ll 1.
\end{aligned}
$$

On the other hand, we have by definition $\varepsilon(0) = -(1-\alpha\chi_B)b_0\eta_{b_0} = -(1-\alpha)b_0\eta_{b_0} - \alpha(1-\chi_B)b_0\eta_{b_0}$ and hence:

$$
\begin{aligned}
\|\mathcal{H}_{b_0}\varepsilon(0)\|_b^2 \lesssim & b_0^2\int_{y\geq B} |b_0\lambda_{b_0}\eta_{b_0}|^2\rho_{b_0}y\,dy + |1-\alpha|^2 b_0^4\lambda_{b_0}^2\|\eta_{b_0}\|_{b_0}^2 \\
&+ b_0^2\int_{B\leq y\leq 2B}\left[\frac{|\partial_y\eta_{b_0}|^2}{y^2} + \frac{\eta_{b_0}^2}{y^4} + b_0^2\eta_{b_0}^2\right]\rho_{b_0}y\,dy.
\end{aligned}
$$

Using (C.2) we can estimate the first term:

$$
b_0^2\int_{y\geq B} |b_0\lambda_{b_0}\eta_{b_0}|^2\rho_{b_0}y dy \lesssim b_0^4\lambda_{b_0}^2 \lesssim b_0^4\lambda_{b_0}^2\frac{1}{b_0|\log b_0|^2} \lesssim \frac{b_0^3}{|\log b_0|^4},
$$

For the second term we use (C.3) and (2.78) and readily obtain

$$
|1-\alpha|^2 b_0^4\lambda_{b_0}^2\|\eta_{b_0}\|_{b_0}^2 \lesssim \frac{b_0^3}{|\log b_0|^8}.
$$

Similarily using the decomposition (2.77) (with $k=0$), we have

$$\begin{aligned}
b_0^2\int_{B\le y\le 2B}\frac{|\partial_y\eta_{b_0}|^2}{y^2}\rho_{b_0}y\,dy &\lesssim b_0^2\int_{B\le y\le 2B}\frac{1}{y^4}\rho_{b_0}y\,dy + b_0^2\int_{B\le y\le 2B}\frac{|\partial_y\eta_{b_0,1}|^2}{y^2}\rho_{b_0}y\,dy\\
&\lesssim b_0^2B^{-4}e^{-b_0B^2/2} + b_0^2B^{-2}\|\partial_y\eta_{b_0,1}\|_{b_0}^2\\
&\lesssim \frac{b_0^4}{|\log b_0|^4} + \frac{b_0^3}{|\log b_0|^4} \lesssim \frac{b_0^3}{|\log b_0|^4},
\end{aligned}$$

where we used (2.78) in the last line. In a similar fashion

$$\begin{aligned}
b_0^2\int_{B\le y\le 2B}\frac{\eta_{b_0}^2}{y^4}\rho_{b_0}y\,dy &\lesssim b_0^2\int_{B\le y\le 2B}\frac{|\log y|^2}{y^4}\rho_{b_0}y\,dy + b_0^2\int_{B\le y\le 2B}\frac{\eta_{b_0,1}^2}{y^4}\rho_{b_0}y\,dy\\
&\lesssim b_0^2B^{-4}|\log B|^2\int_{B\le y\le 2B}\rho_{b_0}y\,dy + b_0^2B^{-4}\|\eta_{b_0,1}\|_{b_0}^2\\
&\lesssim \frac{b_0^3}{|\log b_0|^2}e^{-b_0B^2/2} + \frac{b_0^3}{|\log b_0|^6}\\
&\lesssim \frac{b_0^3}{|\log b_0|^2}e^{-|\log b_0|/4} + \frac{b_0^3}{|\log b_0|^6} \lesssim \frac{b_0^3}{|\log b_0|^6}
\end{aligned}$$

if b_0 is sufficiently small. Finally,

$$\begin{aligned}
b_0^2\int_{B\le y\le 2B} b_0^2\eta_{b_0}^2\rho_{b_0}y\,dy &\lesssim b_0^4\int_{B\le y\le 2B}(\log y)^2\rho_{b_0}y\,dy + b_0^4\|\eta_{b_0,1}\|_{b_0}^2\\
&\lesssim b_0^4|\log B|^2\frac{1}{b_0}e^{-b_0B^2/2} + b_0^4\frac{1}{b_0|\log b_0|^2}\\
&\lesssim b_0^3|\log b_0|^2e^{-\frac{|\log b_0|}{4}} + b_0^3\frac{1}{|\log b_0|^2} \lesssim \frac{b_0^3}{|\log b_0|^2}.
\end{aligned}$$

and hence (3.15) is satisfied. Moreover,

$$(\varepsilon(0),\eta_{b_0})_{b_0} = (-(1-\alpha\chi_B)b_0\eta_{b_0},\eta_{b_0})_{b_0} = -b_0\|\eta_{b_0}\|_{b_0}^2 + \alpha b_0(\chi_{B_0}\eta_{b_0},\eta_{b_0}) = 0$$

by (C.1), and therefore the orthogonality condition from (3.14) is satisfied.

Remark C.1. Observe that by our construction, the initial temperature u_0 is non-negative in Ω. In this case, the solution $u(t,\cdot)$ remains non-negative by the maximum principle.

Appendix D. Cauchy theory in $\dot H^1\times\dot H^2$

Theorem D.1 (Well-posedness in H^2). *Let $u_0\in H^2(\Omega(0))$, $\lambda_0>0$, $u_0(\lambda_0)=0$. Then there exists a time $T=T(\|u_0\|_{H^2(\Omega)},\ \lambda_0)>0$, a constant $C>0$, and a solution (u,λ) to the Stefan problem (1.2) on the time interval $[0,T]$ such that*

$$\begin{aligned}
&u\in C([0,T),H^2(\Omega))\cap L^2([0,T),H^3(\Omega)),\\
&u_t\in C((0,T),L^2(\Omega))\cap L^2([0,T],H^1(\Omega)),\\
&\lambda\in C^1([0,T),\mathbb{R}),
\end{aligned}\tag{D.1}$$

and the following bounds hold:

$$\|u\|_{H^2(\Omega(t))}\le C = C(\|u_0\|_{H^2(\Omega(0)),\lambda_0}),\quad \lambda(t)>\frac{\lambda_0}{2}$$

for some universal polynomial function C of the initial data. Moreover, if $\mathcal{T}$ is the maximal time of existence of a solution (w, λ) satisfying (D.1), *then*

$$\textit{either} \quad \lim_{t \to \mathcal{T}^-} \|u(t,\cdot)\|_{H^2(\Omega(t))} = \infty \quad \textit{or} \quad \lim_{t \to \mathcal{T}^-} \lambda(t) = 0.$$

Remark D.2. The Stefan problem allows for an instant smoothing effect. It is well-known that the solution u becomes infinitely smooth on $(0, \mathcal{T})$ in both the time- and the space variable, see for instance [26, 39, 25].

The proof of Theorem D.1 is presented at the very end of this section, as a simple consequence of Theorem D.5.

We start by pulling-back the problem (1.2) onto the fixed domain $\Omega := \{\mathbf{y} \in \mathbb{R}^2,\ |\mathbf{y}| \geq 1\}$. We denote the points in Ω by bold $\mathbf{y}$, while the radial coordinate $|\mathbf{y}|$ is denoted by y. We define the pull-back temperature function $w : \Omega \to \mathbb{R}$ by

$$w(t, y) = u(t, \lambda(t) y)$$

A simple application of the chain rule gives the following system of equations for w :

$$w_t - \frac{\dot\lambda}{\lambda}\Lambda w - \frac{1}{\lambda^2}\Delta w = 0 \qquad \text{in } \Omega\,; \tag{D.2a}$$

$$w_y(t,1) = -\dot\lambda(t)\lambda(t); \tag{D.2b}$$

$$w(t,1) = 0, \tag{D.2c}$$

$$w(0,\cdot) = w_0\,,\ \lambda(0) = \lambda_0. \tag{D.2d}$$

Lemma D.3 (Energy identities)**.** *Assume that (w, λ) is a smooth solution to the Stefan problem* (D.2) *on some interval* $[0, T]$. *Assume that $\lambda(0) > 0$, $w_0\big|_{y=1} = 0$, and that $w(t,\cdot) \in H^2(\Omega)$ for $t \in [0, T]$. Then on the interval $[0, T]$ the following energy identities hold:*

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \frac{1}{\lambda^2}\|\nabla w\|^2_{L^2(\Omega)} = -\frac{\dot\lambda}{\lambda}\|w\|^2_{L^2(\Omega)}, \tag{D.3}$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2_{L^2(\Omega)} + \frac{1}{\lambda^2}\|\Delta w\|^2_{L^2(\Omega)} = \pi\lambda\dot\lambda^3, \tag{D.4}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta w\|^2_{L^2(\Omega)} &+ \frac{2\pi}{3}\frac{d}{dt}\left(\lambda|\dot\lambda|\right)^3 + \frac{1}{\lambda^2}\|\nabla\Delta w\|^2_{L^2(\Omega)} \\ &= \frac{2\dot\lambda}{\lambda}\|\Delta w\|^2_{L^2(\Omega)} + \pi\dot\lambda^5\lambda^3.\end{aligned} \tag{D.5}$$

Proof. Multiply (D.2a) by w and integrate over Ω. We obtain

$$\begin{aligned}0 &= \frac{1}{2}\frac{d}{dt}\int_\Omega w^2\,d\mathbf{y} - \frac{\dot\lambda}{\lambda}\int_\Omega \Lambda w w\,d\mathbf{y} + \frac{1}{\lambda^2}\|\nabla w\|^2_{L^2(\Omega)} \\ &= \frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} - \frac{\dot\lambda}{2\lambda}\int_1^\infty y^2\partial_y(w^2)\,dy + \frac{1}{\lambda^2}\|\nabla w\|^2_{L^2(\Omega)} \\ &= \frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \frac{\dot\lambda}{\lambda}\|w\|^2_{L^2(\Omega)} + \frac{1}{\lambda^2}\|\nabla w\|^2_{L^2(\Omega)},\end{aligned}$$

which is precisely (D.3). Multiply (D.2a) by $-\Delta w$ and integrate by parts. We obtain

$$
\begin{aligned}
0 &= \int_\Omega \nabla w \cdot \nabla w_t \, d\mathbf{y} - 2\pi(\partial_n w w_t)\big|_{y=1} + \frac{\dot\lambda}{\lambda}\int_\Omega \Lambda w \Delta w \, d\mathbf{y} + \frac{1}{\lambda^2}\|\Delta w\|^2_{L^2(\Omega)} \\
&= \frac{1}{2}\frac{d}{dt}\|\nabla w\|^2_{L^2(\Omega)} + 2\pi(w_y w_t)\big|_{y=1} + 2\pi\frac{\dot\lambda}{\lambda}\int_1^\infty \Lambda w \partial_y \Lambda w \, dy + \frac{1}{\lambda^2}\|\Delta w\|^2_{L^2(\Omega)} \\
&= \frac{1}{2}\frac{d}{dt}\|\nabla w\|^2_{L^2(\Omega)} - \pi\frac{\dot\lambda}{\lambda}(\Lambda w)^2\big|_{y=1} + \frac{1}{\lambda^2}\|\Delta w\|^2_{L^2(\Omega)} \\
&= \frac{1}{2}\frac{d}{dt}\|\nabla w\|^2_{L^2(\Omega)} - \pi\lambda\dot\lambda^3 + \frac{1}{\lambda^2}\|\Delta w\|^2_{L^2(\Omega)},
\end{aligned}
$$

where we used the fact that $w_t(t,1) = 0$, $\Delta w = \frac{1}{y}\partial_y \Lambda w$, and $\Lambda w\big|_{y=1} = w_y\big|_{y=1} = -\lambda\dot\lambda$. Note also that $\partial_n w|_{y=1} = -\partial_y w|_{y=1}$.This proves (D.4). To prove (D.5) we apply ∇ to (D.2), multiply by $-\nabla\Delta w$ and integrate-by-parts. We obtain

$$
\begin{aligned}
0 &= -\int_\Omega \nabla w_t \cdot \nabla\Delta w \, d\mathbf{y} + \frac{\dot\lambda}{\lambda}\int_\Omega \nabla\Lambda w \cdot \nabla\Delta w \, d\mathbf{y} + \frac{1}{\lambda^2}\|\nabla\Delta w\|^2_{L^2(\Omega)} \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega (\Delta w)^2 \, d\mathbf{y} + 2\pi(\partial_y w_t \Delta w)\big|_{y=1} - \frac{\dot\lambda}{\lambda}\int_\Omega \Delta\Lambda w \cdot \Delta w \, d\mathbf{y} \\
&\quad - 2\pi\frac{\dot\lambda}{\lambda}(\partial_y \Lambda w \Delta w)\big|_{y=1} + \frac{1}{\lambda^2}\|\nabla\Delta w\|^2_{L^2(\Omega)}.
\end{aligned} \tag{D.6}
$$

From (D.2b) it follows that

$$
\partial_y w_t\big|_{y=1} = -\partial_t(\lambda\dot\lambda)
$$

Restricting (D.2a) to $y = 1$ we conclude that $-\frac{\dot\lambda}{\lambda}w_y\big|_{y=1} - \frac{1}{\lambda^2}(\Delta w)\big|_{y=1} = 0$. Using (D.2b) this implies that

$$
(\Delta w)\big|_{y=1} = \lambda^2\dot\lambda^2. \tag{D.7}
$$

The previous two boundary identities imply that

$$
2\pi(\partial_y w_t \Delta w)\big|_{y=1} = -\frac{2\pi}{3}\frac{d}{dt}\left(\lambda\dot\lambda\right)^3. \tag{D.8}
$$

Note that $\partial_y \Lambda w = w_y + y w_{yy}$ and therefore, when restricted to $y = 1$ we conclude that $(\partial_y \Lambda w)\big|_{y=1} = (\Delta w)\big|_{y=1}$. From (D.7) we infer that

$$
(\partial_y \Lambda w)\big|_{y=1} = \lambda^2\dot\lambda^2.
$$

Therefore

$$
-2\pi\frac{\dot\lambda}{\lambda}(\partial_y \Lambda w \Delta w)\big|_{y=1} = -2\pi\lambda^3\dot\lambda^5. \tag{D.9}
$$

It remains to evaluate the term $-\frac{\dot\lambda}{\lambda}\int_\Omega \Delta\Lambda w \cdot \Delta w \, d\mathbf{y}$. A direct calculation yields

$$
\Delta\Lambda w = \Lambda\Delta w + 2\Delta w.
$$

Therefore

$$
\begin{aligned}
-\frac{\dot\lambda}{\lambda}\int_\Omega \Delta\Lambda w \cdot \Delta w \, d\mathbf{y} &= -\frac{2\dot\lambda}{\lambda}\|\Delta w\|^2_{L^2(\Omega)} - 2\pi\frac{\dot\lambda}{\lambda}\int_1^\infty \partial_y \Delta w \Delta w \, dy \\
&= -\frac{2\dot\lambda}{\lambda}\|\Delta w\|^2_{L^2(\Omega)} + \pi\frac{\dot\lambda}{\lambda}(\Delta w)^2\big|_{y=1}.
\end{aligned} \tag{D.10}
$$

Plugging (D.8), (D.9), and (D.10) into (D.6), we obtain (D.5). □

Let us define the energy-like quantities

$$E(t) = \sup_{0\le s\le t}\left\{\frac{1}{2}\|w(s,\cdot)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla w(s,\cdot)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Delta w(s,\cdot)\|^2_{L^2(\Omega)}\right\}, \quad \text{(D.11)}$$

$$D(t) = \frac{1}{\lambda(t)^2}\|\nabla w(t,\cdot)\|^2_{L^2(\Omega)} + \frac{1}{\lambda(t)^2}\|\Delta w(t,\cdot))\|^2_{L^2(\Omega)} + \frac{1}{\lambda(t)^2}\|\nabla\Delta w(t,\cdot)\|^2_{L^2(\Omega)}. \quad \text{(D.12)}$$

Lemma D.4 (A priori estimate). *Assume that* (w,λ) *is a smooth solution to* (D.2) *on some interval* $[0,T^*]$. *Assume that* $\lambda_0 > 0$, $w_0\big|_{y=1} = 0$, *and that* $w(t,\cdot)\in H^2(\Omega)$ *for* $t\in[0,T^*]$. *Then there exists a* $T = T(E(0),\lambda_0) > 0$, $T\le T^*$, *such that for any* $t\in[0,T]$ *the following a priori bounds hold:*

$$E(t) \le 4E(0), \quad \text{(D.13)}$$

$$\lambda(t) > \frac{\lambda_0}{2}. \quad \text{(D.14)}$$

Proof. From Lemma D.3 we infer that

$$\begin{aligned} E(t) + \int_0^t D(s)\,ds \le & E(0) + \frac{2\pi}{3}\lambda_0^3\dot\lambda(0)^3 - \frac{2\pi}{3}\lambda(t)^3\dot\lambda(t)^3 + \int_0^t \frac{|\dot\lambda(s)|}{\lambda(s)}\,ds\,E(t) \\ & + \pi\int_0^t\left(\lambda^3(s)|\dot\lambda(s)|^5 + \lambda(s)\dot\lambda(s)^3\right)ds \end{aligned} \quad \text{(D.15)}$$

Note that

$$w_y^2\big|_{y=1} \lesssim \|\Delta w\|^2_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)}.$$

Therefore

$$|\dot\lambda(t)|^2 \le \frac{C}{\lambda^2}E(t), \quad C > 1. \quad \text{(D.16)}$$

Using (D.16) we obtain

$$\begin{aligned} & E(t)\int_0^t \frac{|\dot\lambda(s)|}{\lambda(s)}\,ds + \pi\int_0^t\left(\lambda^3(s)|\dot\lambda(s)|^5 + \lambda(s)|\dot\lambda(s)|^3\right)ds \\ & \le C't\left(E^2(t) + E^{3/2}(t) + E^{5/2}(t)\right)\sup_{0\le s\le t}\frac{1}{\lambda^2(s)}. \end{aligned} \quad \text{(D.17)}$$

To bound the error term $-\frac{2\pi}{3}\lambda(t)^3\dot\lambda(t)^3$ we need a more refined estimate than (D.16) due to the absence of the integral-in-time. Note that by the trace inequality and the interpolation between fractional Sobolev spaces, we have

$$|w_y(1)| \lesssim \|\nabla w\|_{H^{1/2}(\Omega)} \lesssim \|\nabla w\|^{1/2}_{L^2(\Omega)}\|\nabla w\|^{1/2}_{H^1(\Omega)}.$$

Therefore, upon using the Young inequality

$$\begin{aligned} |\lambda\dot\lambda|^3 & \lesssim \|\nabla w\|^{3/2}_{L^2(\Omega)}\|\nabla w\|^{3/2}_{H^1(\Omega)} \le \delta\|\nabla w\|^2_{H^1(\Omega)} + C_\delta\|\nabla w\|^6_{L^2(\Omega)} \\ & \le \delta E + C_\delta\|\nabla w\|^6_{L^2(\Omega)}. \end{aligned} \quad \text{(D.18)}$$

Integrating (D.4) over $[0,t]$, we have

$$\begin{aligned} \sup_{0\le s\le t}\|\nabla w\|^2_{L^2(\Omega)} & \le \|\nabla w_0\|^2_{L^2(\Omega)} + \pi\int_0^t \lambda(s)|\dot\lambda(s)|^3\,ds \\ & \le E_0 + C^*tE^{3/2}(t)\sup_{0\le s\le t}\frac{1}{\lambda^2(s)}. \end{aligned}$$

Therefore, we obtain from (D.18) that

$$|\lambda\dot{\lambda}|^3 \leq C_0 + \delta E + Ctp(E)q(\sup_{0\leq s\leq t}\frac{1}{\lambda^2(s)})$$

where p and q are increasing polynomial functions of their arguments. Plugging this bound back into (D.15), using (D.17), and the definition of $E(t)$ we conclude that

$$E(t) + \int_0^t D(s)\,ds \leq C_0 + Ctp(E(t))q(\sup_{0\leq s\leq t}\frac{1}{\lambda^2(s)}), \tag{D.19}$$

where $C_0 = C_0(E(0), \lambda_0)$. Since $\lambda(t) = \lambda(0) + \int_0^t \dot{\lambda}(s)\,ds$, it follows that

$$\lambda(t) \geq \lambda_0 - t\sup_{0\leq s\leq t}|\dot{\lambda}(s)| \geq \lambda_0 - \frac{Ct}{\lambda}E^{1/2}(t).$$

Let

$$T' = \sup\{t \geq 0 \big| E(t) \leq 4E(0),\ \lambda(t) > \lambda_0/2\}.$$

By the continuity of $E(\cdot)$ and $\lambda(\cdot)$ it follows that $T' > 0$. On $[0, T']$ we therefore have

$$E(t) + \int_0^t D(s)\,ds \leq C_0 + Ctp(E(t))q(\frac{1}{\lambda_0 - 4\frac{\sqrt{C}}{\lambda(0)}tE(0)^{1/2}}) \tag{D.20}$$

By a standard continuity argument, there exists a sufficiently small $T = T(E(0), \lambda_0)$, $T \leq \frac{\lambda_0}{16\sqrt{C}E(0)^{1/2}}$ such that

$$E(t) \leq 2C_0, \quad t \in [0, T].$$

By the choice of T, we also have the bound $\lambda(t) \geq \frac{3}{4}\lambda_0 > \frac{1}{2}\lambda_0$ for $t \in [0, T]$ and this concludes the proof of the lemma.

□

Theorem D.5 (Local well-posedness)**.** *Let $w_0 \in H^2(\Omega)$, $\lambda_0 > 0$, and $w_0\big|_{y=1} = 0$. Then there exists a time $T = T(\|w_0\|_{H^2(\Omega)},\ \lambda(0)) > 0$ and a solution (w, λ) to the Stefan problem* (D.2) *on the time interval $[0, T]$ such that*

$$\begin{aligned}
&w \in C([0, T], H^2(\Omega)) \cap L^2([0, T], H^3(\Omega)),\\
&w_t \in C((0, T], L^2(\Omega)) \cap L^2([0, T], H^1(\Omega)),\\
&\lambda \in C^1([0, T], \mathbb{R}),
\end{aligned} \tag{D.21}$$

satisfying the energy estimate

$$E(t) \leq C_0 = C_0(E(0)), \quad t \in [0, T]$$

and the lower bound

$$\lambda(t) > \frac{\lambda(0)}{2} \quad t \in [0, T],$$

where the energy $E(\cdot)$ is defined by (D.11)*. Moreover, if $\mathcal{T}$ is the maximal time of existence of a solution (w, λ) satisfying* (D.21)*, then*

$$\text{either} \quad \lim_{t\to\mathcal{T}^-}\|w(t,\cdot)\|_{H^2(\Omega)} = \infty \quad \text{or} \quad \lim_{t\to\mathcal{T}^-}\lambda(t) = 0.$$

Proof. The proof of existence follows a standard iteration argument for the sequence of approximations $(\lambda_n(t), w_n(t))$, $n \in \mathbb{N}$. For a given $\lambda_n(\cdot)$ we define w_{n+1} by solving

$$\partial_t w_{n+1} - \frac{\partial_t\lambda_n}{\lambda_n}\Lambda w_{n+1} - \frac{1}{\lambda_n^2}\Delta w_{n+1} = 0 \ \text{ in } \Omega\,; \tag{D.22a}$$

$$w_{n+1}(t, 1) = 0.$$

We then update $\lambda_{n+1}(\cdot)$ by solving

$$\partial_y w_{n+1}(t,1) = -\partial_t \lambda_{n+1}(t)\lambda_n(t).$$

Estimates analogous to the a priori estimates of Lemma D.4 can be used to obtain uniform bounds on $E(w_n,\lambda_n) + \int_0^t D(w_n,\lambda_n)\,ds$, where E and D are defined by (D.11) and (D.12) respectively. Note that we can also get uniform bounds on $\|\partial_t w_n\|_{L^\infty([0,T],L^2(\Omega))} + \|\partial_t w_n\|_{L^2([0,T],H^1(\Omega))}$ as the latter norms are controlled by $E(w_n,\lambda_n) + \int_0^t D(w_n,\lambda_n)\,ds$ from (D.22a). Upon passing to the limit, we obtain a solution to which the energy estimate of Lemma D.4 applies. The proof of uniqueness is standard. The breakdown criterion is a simple consequence of (D.11), and (D.16). $\square$

Proof of Theorem D.1. Let w be the solution to (D.2) as given by Theorem D.5. It is easy to check that $\|w(t,\cdot)\|_{H^2(\Omega)} \lesssim E(t)$, $t \in [0,\mathcal{T})$. Theorem D.1 now follows from the change of variables $u(t,r) = w(t,\frac{r}{\lambda(t)})$ and Theorem D.5.

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